

**Ya. Shnir**  
**Magnetic Monopoles**


This monograph addresses the field theoretical aspects of magnetic monopoles. Written for graduate students as well as researchers, the author demonstrates the interplay between mathematics and physics. He delves into details as necessary and develops many techniques that find applications in modern theoretical physics. This introduction to the basic ideas used for the description and construction of monopoles is also the first coherent presentation of the concept of magnetic monopoles. It arises in many different contexts in modern theoretical physics, from classical mechanics and electrodynamics to multidimensional branes. The book summarizes the present status of the theory and gives an extensive but carefully selected bibliography on the subject. The first part deals with the Dirac monopole, followed in part two by the monopole in non-Abelian gauge theories. The third part is devoted to monopoles in supersymmetric Yang-Mills theories.

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## Part I Dirac Monopole

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<b>1</b>	<b>Magnetic Monopole in Classical Theory</b> .....	<b>3</b>
1.1	Non-Relativistic Scattering on a Magnetic Charge .....	3
1.2	Non-Relativistic Scattering on a Dyon .....	10
1.3	Vector Potential of a Monopole Field .....	12
1.4	Transformations of the String .....	15
1.5	Dynamical Symmetries of the Charge-Monopole System .....	20
1.6	Dual Invariance of Classical Electrodynamics .....	22
<b>2</b>	<b>The Electron–Monopole System: Quantum-Mechanical Interaction</b> .....	<b>27</b>
2.1	Charge Quantization Condition .....	27
2.2	Spin-Statistics Theorem in a Monopole Theory .....	31
2.3	Charge-Monopole System: Quantum-Mechanical Description .	33
2.3.1	The Generalized Spherical Harmonics .....	34
2.3.2	Solving the Radial Schrödinger Equation .....	37
2.4	Non-Relativistic Scattering on a Monopole: Quantum Mechanical Description .....	40
2.5	Charge-Monopole System: Spin in the Pauli Approximation ..	42
2.5.1	Dynamical Supersymmetry of the Electron-Monopole System .....	44
2.5.2	Generalized Spinor Harmonics: $j \geq \mu + 1/2$ .....	46
2.5.3	Generalized Spinor Harmonics: $j = \mu - 1/2$ .....	48
2.5.4	Solving the Radial Pauli Equation .....	49
2.6	Charge-Monopole System: Solving the Dirac Equation .....	53
2.6.1	Zero Modes and Witten Effect .....	55
2.6.2	Charge Quantization Condition and the Group $SL(2, \mathbb{Z})$ .....	61
<b>3</b>	<b>Topological Roots of the Abelian Monopole</b> .....	<b>67</b>
3.1	Abelian Wu–Yang Monopole .....	67
3.2	Differential Geometry and Topology .....	70
3.2.1	Notions of Topology .....	70
3.2.2	Notions of Differential Geometry .....	81

3.2.3	Maxwell Electrodynamics and Differential Forms . . . . .	89
3.3	Wu–Yang Monopole and the Fiber-Bundle Topology . . . . .	93
3.3.1	Fiber Bundles . . . . .	93
3.3.2	Principal Bundle and Connection . . . . .	97
3.3.3	Wu–Yang Monopole Bundle . . . . .	102
3.3.4	Hopf Bundle . . . . .	103
<b>4</b>	<b>Abelian Monopole:</b>	
	<b>Relativistic Quantum Theory . . . . .</b>	<b>109</b>
4.1	Two Types of Charges . . . . .	110
4.2	Two-Potential Formulation of Electrodynamics . . . . .	112
4.2.1	Energy-Momentum Tensor and Angular Momentum . . . . .	115
4.3	Canonical Quantization . . . . .	118
4.3.1	Relativistic Invariance of Two-Charge Electrodynamics . . . . .	121
4.4	Renormalization of QED with a Magnetic Charge . . . . .	125
4.5	Vacuum Polarization by a Dyon Field . . . . .	128
4.6	Effective Lagrangian of QED with a Magnetic Charge . . . . .	132

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## Part II Monopole in Non-Abelian Gauge Theories

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<b>5</b>	<b>'t Hooft–Polyakov Monopole . . . . .</b>	<b>141</b>
5.1	$SU(2)$ Georgi–Glashow Model and the Vacuum Structure . . . . .	141
5.1.1	Non-Abelian Wu–Yang Monopole . . . . .	141
5.1.2	Georgi–Glashow Model . . . . .	143
5.1.3	Topological Classification of the Solutions . . . . .	146
5.1.4	Definition of Magnetic Charge . . . . .	148
5.1.5	't Hooft–Polyakov Ansatz . . . . .	151
5.1.6	Singular Gauge Transformations and the Connection between 't Hooft–Polyakov and Dirac monopoles . . . . .	154
5.1.7	Dyons . . . . .	155
5.2	The Bogomol'nyi Limit . . . . .	157
5.2.1	Gauge Zero Mode and the Electric Dyon Charge . . . . .	161
5.3	Topological Classification of Non-Abelian Monopoles . . . . .	163
5.3.1	$SO(3)$ vs $SU(2)$ . . . . .	163
5.3.2	Magnetic Charge and the Topology of the Gauge Group . . . . .	165
5.3.3	Equivalence of Topological and Magnetic Charge . . . . .	166
5.3.4	Topology of the Dyon Sector . . . . .	168
5.4	The $\theta$ Term and the Witten Effect Again . . . . .	170

<b>6</b>	<b>Multimonopole Configurations</b> . . . . .	173
6.1	Multimonopoles Configurations and Singular Gauge Transformations . . . . .	174
6.1.1	Singular $SU(2)$ Monopole with Charge $g = ng_0$ . . . . .	174
6.1.2	Magnetic Dipole . . . . .	176
6.2	Rebbi–Rossi Multimonopoles, Chains of Monopoles and Closed Vortices . . . . .	178
6.3	Interaction of Magnetic Monopoles . . . . .	192
6.3.1	Monopole in External Magnetic Field . . . . .	192
6.3.2	The Interaction Energy of Monopoles . . . . .	194
6.3.3	Classical Interaction of Two Widely Separated Dyons . . . . .	197
6.4	The $n$ -Monopole Configuration in the BPS Limit . . . . .	201
6.4.1	BPS Multimonopoles: A Bird’s Eye View . . . . .	201
6.4.2	Projective Spaces and Twistor Methods . . . . .	203
6.4.3	The $n$ -Monopole Twistor Construction . . . . .	206
6.4.4	Hitchin Approach and the Spectral Curve . . . . .	214
6.4.5	Nahm Equations . . . . .	217
6.4.6	Solution of the Nahm Equations . . . . .	220
6.4.7	The Nahm Data and Spectral Curve . . . . .	223
6.5	Moduli Space and Low-Energy Multimonopoles Dynamics . . . . .	227
6.5.1	Zero Modes Lagrangian and the Moduli Space Metric . . . . .	227
6.5.2	Metric on the Space $\mathcal{M}_2$ . . . . .	232
6.5.3	Low-Energy Scattering of Two Monopoles . . . . .	236
<b>7</b>	<b><math>SU(2)</math> Monopole in Quantum Theory</b> . . . . .	241
7.1	Field Fluctuations on Monopole Background . . . . .	241
7.1.1	Generalized Angular Momentum and the Spectrum of Fluctuations . . . . .	245
7.1.2	Quantum Correction to the Mass of a Monopole . . . . .	250
7.2	Non-Abelian Monopole: Quasiclassical Quantization . . . . .	254
7.2.1	Collective Coordinates and Constraints . . . . .	254
7.2.2	Quantum Mechanics on the Moduli Space . . . . .	258
7.2.3	Evaluation of the Generating Functional . . . . .	263
7.3	$g\bar{g}$ Pair Creation in an External Magnetic Field . . . . .	267
7.3.1	Dynamics of Non-Abelian Monopole in Weak External Field . . . . .	267
7.3.2	Metastable Vacuum Decay and Monopole Pair Creation in an External Field . . . . .	269
<b>8</b>	<b>Monopoles Beyond <math>SU(2)</math> Group</b> . . . . .	275
8.1	$SU(N)$ Monopoles . . . . .	276
8.1.1	Generalization of the Charge Quantization Condition . . . . .	276
8.1.2	Towards Higher Rank Gauge Groups . . . . .	277
8.1.3	Montonen–Olive Conjecture . . . . .	279

8.1.4	Cartan–Weyl Basis and the Simple Roots	282
8.1.5	$SU(3)$ Cartan Algebra	284
8.1.6	$SU(3)$ Monopoles	287
8.2	Massive and Massless Monopoles	301
8.2.1	Pathologies of Non-Abelian Gauge Transformations	303
8.3	$SU(3)$ Monopole Moduli Space	306
8.3.1	$SU(3)$ Monopoles: Nahm Equations	314
<b>9</b>	<b>Monopoles and the Problem of Confinement</b>	<b>319</b>
9.1	Quark Confinement in QCD	319
9.1.1	Dual Superconductor	324
9.2	Monopoles in the Lattice QCD	328
9.2.1	Compact QED and Lattice Monopoles	331
9.2.2	Lattice Duality	334
9.3	Abelian Projection	339
9.3.1	“Monopoles” from Abelian Projection	339
9.3.2	Maximal Abelian Gauge	346
9.4	Polyakov Solution of Confinement in the $d = 3$ Georgi–Glashow Model	349
9.4.1	Dilute Gas of Monopoles in the $d = 3$ Georgi–Glashow Model	349
9.4.2	Wilson Loop Operator in $d = 3$ Georgi–Glashow Model	355
<b>10</b>	<b>Rubakov–Callan Effect</b>	<b>359</b>
10.1	Dirac Hamiltonian on the Non-Abelian Monopole Background	359
10.1.1	Fermionic Zero Modes	363
10.1.2	Zero Modes and the Index Theorem	367
10.1.3	$S$ -Wave Fermion Scattering on a Monopole	373
10.2	Anomalous Non-Conservation of the Fermion Number	378
10.2.1	Axial Anomaly and the Vacuum Structure	378
10.2.2	Effective Action of Massless Fermions	379
10.2.3	Properties of the Anomalous Fermion Condensate	385
10.2.4	Properties of Other Condensates	388
10.3	Monopole-Fermion Scattering in the Bosonisation Technique	390
10.3.1	Vertex Operator and Bosonization of the Free Model	391
10.3.2	Monopole Catalysis of the Proton Decay	397
10.3.3	Monopole Catalysis of the Proton Decay: Semiclassical Model	400

<b>11</b>	<b>Supersymmetric Yang-Mills Theories</b> .....	407
11.1	What is Supersymmetry? .....	407
11.1.1	Poincaré Group and Algebra of Generators .....	408
11.1.2	Algebra of Generators of Supersymmetry .....	412
11.2	Representations of SUSY Algebra .....	415
11.2.1	$N = 1$ Massive Multiplets .....	415
11.2.2	$N = 1$ Massless Multiplets .....	417
11.2.3	$N = 2$ Extended SUSY .....	418
11.3	Local Representations of SUSY .....	420
11.3.1	$N = 1$ Superspace .....	420
11.3.2	$N = 1$ Superfields .....	424
11.3.3	Non-Abelian Multiplets .....	428
11.4	$N = 1$ SUSY Lagrangians .....	429
<b>12</b>	<b>Magnetic Monopoles in the <math>N = 2</math> Supersymmetric Yang-Mills Theory</b> .....	437
12.1	$N = 2$ Supersymmetric Lagrangian .....	437
12.1.1	Praise of Beauty of $N = 2$ SUSY Yang-Mills .....	441
12.2	$N = 2$ Supersymmetric $SU(2)$ Magnetic Monopoles .....	443
12.2.1	Construction of $N = 2$ Supersymmetric SU(2) Monopoles .....	443
12.3	Central Charges in the $N = 2$ SUSY Yang-Mills .....	446
12.4	Fermionic Zero Modes in Supersymmetric Theory .....	449
12.5	Low Energy Dynamics of Supersymmetric Monopoles .....	451
12.6	$N = 2$ Supersymmetric Monopoles beyond $SU(2)$ .....	453
12.6.1	$SU(3)$ $N = 2$ Supersymmetric Monopoles .....	458
<b>13</b>	<b>Seiberg-Witten Solution of <math>N = 2</math> SUSY Yang-Mills Theory</b> .....	465
13.1	Moduli Space .....	466
13.1.1	Moduli Space and its Parameterization .....	466
13.1.2	Quantum Moduli Space of $N = 2$ SUSY Yang-Mills Theory .....	472
13.2	Global Parametrization of the Quantum Moduli Space .....	478
13.2.1	Transformation of Duality for $N = 2$ Low-Energy Effective Theory .....	478
13.2.2	BPS Bound Reexamined .....	483
13.3	Seiberg-Witten Explicit Solution .....	485
13.3.1	Monodromies on the Moduli Space .....	485
13.3.2	Solution of the Monodromy Problem .....	492
13.3.3	Confinement and the Monopole Condensation .....	496
13.4	Concluding Remarks .....	498

XVIII Contents

<b>A</b>	<b>Representations of <math>SU(2)</math></b> .....	501
<b>B</b>	<b>Quaternions</b> .....	505
<b>C</b>	<b><math>SU(2)</math> Transformations of the Monopole Potential</b> .....	509
	<b>References</b> .....	513
	<b>Index</b> .....	529