Contents

1. Sets, Sequences, Series, and Functions
   - Basic Set Definitions....1
   - Unions, Intersections (multiple)....4
   - Lim Inf, Lim Sup, Limit, Convergence of Set Sequences..4
   - Sup, Inf, Max, Min of Real Sets....9
   - Limit Point of Real Sets; Closure, Boundary....11
   - Open, Closed Real Sets....12
   - Bolzano-Weierstrass Theorem....14
   - Limit Point of Real Sequences....15
   - Lim Inf, Lim Sup, Limit, Convergence of Real Sequences16
   - Cauchy Criterion for Convergence....20
   - Sup, Inf, Max, Min of Functions over Sets....22
   - General Principle of Convergence for Real Series....28
   - Properties of Convergent Series....29
   - Bracketing and Reordering....29
   - Non-negative Series, Absolute Convergence....30
   - Tests for Convergence of Real Series....33
   - Hints and Answers....44
   - References....61

2. Doubly Infinite Sequences and Series
   - Definitions and Notation: Sequences....63
   - Lim Inf, Lim Sup, Limit, Convergence: Sequences....64
   - Cauchy Criterion: Sequences....66
   - Iterated Limits: Sequences....66
   - Definition and Notation: Series....71
   - Iterated Sums: Series....72
   - Convergence: Series....73
   - Non-Negative Series....76
   - Absolute Convergence: Series....79
   - Tests for Convergence: Series....80
   - Interchange of Summation Order: Series....82
   - Hints and Answers....86
   - References....93

3. Sequences and Series of Functions
   - Real Function Sequences: Definition, Notation....95
   - Lim Inf, Lim Sup, Pointwise Convergence, Limit....95
   - Pointwise Convergence: Shortcomings....98
   - Uniform Convergence: Real Function Sequences....98
   - Continuity of Limit Under Uniform Convergence....102
   - Real Function Sequences: Monotone, Continuous....104
   - Term-by-Term Integration....107
   - Term-by-Term Differentiation....109
   - Real Function Series: Definition, Notation....110
   - Sum Function and Pointwise Convergence....111
   - Interchanging Limit Operations: Dominated Convergence....112
   - Interchanging Limit Operations: Fatou's Lemma....116

Uniform Convergence: Real Function Series.....117
Real Function Series: Uniform Convergence Tests.....118
Continuity of Sum Function.....123
Term-by-Term Integration.....123
Term-by-Term Differentiation.....125
Multiply Infinite Case.....126
Hints and Answers.....127
References.....132

4. Real Power Series
Real Power Series about a Point.....133
Radius of Convergence.....133
Convergence.....134
Uniform Convergence of Real Power Series.....135
Interval of Convergence.....136
Continuity of Sum Function.....136
Term-by-Term Integration.....137
Term-by-Term Differentiation.....138
Taylor Series: Definition.....139
Real Geometric Series.....139
Hints and Answers.....140
References.....141

5. Behavior of a Function Near a Point: Various Types of Limits
Notation: Types of Limits.....143
Two-Sided Limit.....144
Continuity.....146
Left-Hand Limit.....146
Left-Continuity.....147
Right-Hand Limit.....148
Right-Continuity.....149
Extensions.....151
Operations With Limits.....153
L'Hospital's Rules.....156
Limit Infimum: Definition, Properties.....159
Limit Supremum: Definition, Properties.....162
Limit Infimum and Supremum: Combined Properties.....163
Applications: Generalized Inequalities.....165
Hints and Answers.....168
References.....179

6. Orders of Magnitude: The O, o, ~ Notation
Comparing Asymptotic Magnitudes.....181
Same Order of Magnitude: the ~ Relation.....181
At Most Order of Magnitude: the O Relation.....185
Smaller Order of Magnitude: the o Relation.....186
Hints and Answers.....188
References.....190

7. Some Abelian and Tauberian Theorems
The Laplace Transform of a Function.....191
Nature of Abelian and Tauberian Theorems.....192
Classical Results.....193
Reformulations.....194
Functions of Slow and Regular Variation.....195
A General Abelian-Tauberian Theorem.....196
Infinite Series Version.....197
Hints and Answers.....198
References.....200

8. 1-Dimensional Cumulative Distribution Functions and Bounded Variation Functions
1-C.D.F.: Definition, Properties.....201
1-C.D.F.: Riemann-Continuous Case.....202
Functions of 1-C.F.F.'s.....203
Sequences of 1-C.D.F.'s: Complete, Weak Convergence.203
Convergence Properties.....203
1-B.V.F.'s: Definition, Relation to 1-C.D.F.'s.....204
1-B.V.F.'s: Properties.....205
1-B.V.F.'s: Alternate Definition by Variation Sums..206
Combinations of 1-B.V.F.'s.....206
Sequences and Convergences of 1-B.V.F.'s.....208
Hints and Answers.....209
References.....213

9. 1-Dimensional Riemann-Stieltjes Integral
Approximating Sums: Partition of Bounded Interval [a,b).....215
Definition and Notation: Integral with 1-C.D.F. Integrator.....216
Sufficient Conditions for Existence.....217
Integration Over a Single Point.....218
Physical Interpretation of the Integral.....219
Extension of Definition: all Bounded Intervals.....221
Properties of the Integral.....223
Integral Inequalities.....224
A Mean Value Theorem.....225
Extension of Definition: Unbounded Intervals.....225
Limit Properties:
Varying Integrand.....229
Varying Integrator.....230
Varying Integration Interval.....233
Integration-by-Parts:
Version A.....235
Version B.....237
Extension of the Integral to Case of:
1-B.V.F. Integrators.....241
Complex-Valued Integrands.....241
Discontinuous Integrands.....242
Change-of-Variables Formula.....243
Differentiation of the Indefinite Integral.....247
Hints and Answers.....248
References.....253

10. n-Dimensional Cumulative Distribution Functions and Bounded Variation Functions
n-Monotonicity: Definition....255
n-Monotonicity: Characterization in Differentiable Case....256
n-C.D.F.'s: Definition, Properties....257
Functions of n-C.D.F.'s....258
n-C.D.F.'s: Riemann Continuous Case....259
Sequences of n-C.D.F.'s: Complete, Weak Convergence.260
n-B.V.F.'s: Definition, Relation to n-C.D.F.'s....260
n-B.V.F.'s: Case of Differentiability....261
n-B.V.F.'s: Alternate Characterization; Variation Sums....261
Hints and Answers....262
References....264

11. n-Dimensional Riemann-Stieltjes Integral
Approximating Sums: Partition of n-Rectangle [a',b')..265
Definition and Notation: n-C.D.F. Integrator....265
Sufficient Conditions for Existence....266
Extensions of Definition to:
Various Bounded Rectangles....267
Unbounded Rectangles....274
Unions of Rectangles....274
Properties of the Integral....276
A Mean Value Theorem....276
Factoring, Iterated Integrals....277
Limit Properties:
Varying Integrand....279
Varying Integrator....280
Varying Integration Set....280
An Integration-by-Parts Formula....280
Extension of the Integral to Case of:
n-B.V.F. Integrators....281
Complex Valued Integrands....281
Discontinuous Integrands....281
Hints and Answers....282
References....285

12. Finite Differences and Difference Equations
Definition: Δ and E Operators....287
Definition: First Unit Differences....288
Definition: n-th Unit Difference (n > 1)....288
Simple Ascending and Descending Factorials....289
Stirling Numbers: First Kind....290
Stirling Numbers: Second Kind....291
Δ Operator: Properties....293
General Ascending and Descending Factorials....294
Definition: Δ-Inverse Operator....295
Anti-Differences and Properties of Δ-Inverse Operators....296
Anti-Differences: Techniques for Obtaining....297
Application: Summation of Series....300
Definition: Difference Equations....303
n-th Order Linear Difference Equations....303
General Solution: n-th Order Homogeneous; Constant Coefficient....304

n-th Order Non-Homogeneous Case; Constant Coefficients....308
Techniques of Solution....313
Simultaneous Difference Equations....313
Hints and Answers....314
References....318

13. Complex Variables
Basic Definitions....319
Modulus....320
Addition, Multiplication, Division....321
Conjugate....322
Polar Form....324
Function of a Complex Variable....326
Limit at a Point....328
Properties of Limit....329
Differentiability at a Point....330
Cauchy-Riemann Equations....332
Sufficient Condition for Differentiability....335
Regularity....336
Derivatives of Regular Functions....336
Complex Power Series....337
Contours....339
Integral of a Complex Function....340
Properties of the Integral....343
Cauchy's Theorem and Goursat's Lemma....347
Evaluation of Certain Integrals by Cauchy's Theorem..352
Cauchy's Integral Formula....355
Integral Formula for Derivatives of Regular Functions....357
Morera's Theorem: a Converse of Cauchy's Theorem....360
Taylor's Theorem: General Form....362
Comparing Complex and Real Variables....363
Integral Inequality due to Cauchy....364
Liouville's Theorem....365
Fundamental Theorem of Algebra....367
Zeros of a Function....368
Isolated Zeros....368
Poles and Singularities....368
Laurent's Theorem and Expansion....370
Types of Singularities....373
Residues....374
Evaluation of Residues
Method A....374
Method B....375
Fundamental Residue Theorem....377
Applications: Evaluation of Contour Integrals....378
Hints and Answers....382
References....392

14. Matrices and Determinants
Definitions
Addition of Matrices....395
Multiplication of Matrices....396
Transpose of a Matrix....396
Conjugate of a Matrix....397
Determinant of a Square Matrix....397
Submatrix, Minor, Principal Minor, Cofactor....398
Evaluation of a Determinant....398
Properties of a Determinant....400
Inverse: Existence and Uniqueness....401
Special Types of Square Matrices and Their Properties
Singular, Non-Singular....402
Symmetric....405
Hermetian....406
Skew-Symmetric....407
Unitary....408
Normal....409
Orthogonal....410
Hints and Answers....410
References....411
....414

15. Vectors and Vector Spaces
Row Vectors of Complex Numbers
Independence, Dependence of Sets of Row Vectors....417
Vector Space and Vector Subspace....417
Subspace Generated by Rows of a Matrix....418
Basis of a Subspace....419
Row Operations on a Matrix....420
Existence of a Basis....421
Unique Representation in Terms of a Fixed Basis....423
Transformations of Basis Vectors....424
Ranks of Subspaces....425
Inner (dot) Product of Vectors....426
Length of a Vector....428
Orthogonality of Vectors....428
Orthogonal Subspaces....430
Ortho-Normal Basis: the Gram-Schmidt Procedure....431
Conjugate Subspaces....432
Hints and Answers....433
References....434
....436

16. Systems of Linear Equations and Generalized Inverse
m Homogeneous Linear Equations in n(≥ m) Unknowns
General Solution....437
"Sweep Out" Technique for Finding General Solution....438
Vector-Space Interpretation of General Solution....439
Rank of Matrix and Rank of Subspace....442
Properties of Rank....443
m Non-Homogeneous Equations in n(≥ m) Unknowns....445
Consistency and Inconsistency....446
"Sweep Out" Technique for Finding General Solution....447
Vector-Space Interpretation of General Solution....448
Generalized Inverse of a Matrix....451
....453

Special Case of Non-Singular Matrix.....454
Technique for Obtaining a Generalized Inverse.....454
Hints and Answers.....456
References.....459

17. Characteristic Roots and Related Topics
Characteristic Root.....461
Characteristic Vector.....461
Characteristic Polynomial.....462
Characteristic Equation.....462
Determinant and Characteristic Roots.....463
Characteristic Roots of:
Markov Matrices.....464
Transpose and Conjugate.....464
Similar Matrices.....465
Inverse.....465
Scalar Multiple.....466
Triangular Matrices.....466
Real, Symmetric Matrices.....466
Trace.....467
Characteristic Roots and Trace.....467
Characteristic Roots of AB and BA.....468
Characteristic Roots of Powers of a Matrix.....468
Ortho-Normal Characteristic Vectors.....469
Characteristic Roots of Orthogonal and Unitary Matrices.....470
Representation Theorem: Real, Symmetric Matrices....471
Rank and Characteristic Roots.....473
Real Quadratic Forms.....474
Positive Definite, Positive Semi-Definite Forms.....475
Principal Axes Theorem.....476
Inverse of a P.D. Matrix.....478
Characteristic Roots and P.D., P.S.D. Matrices.....479
Submatrices of a P.D., P.S.D. Matrix.....479
Test for P.D. or P.S.D. Matrix.....480
Characterization of P.D. Matrices.....481
Largest Characteristic Root.....482
Applications of Largest Characteristic Root.....483
Hints and Answers.....484
References.....490

18. Convex Sets and Convex Functions
Definition: Convex Set in E_n.....491
Convex Linear Combination of Points in E_n.....492
Convex Hull.....493
Inner Product of Points in E_n.....494
Hyperplane in E_n.....494
Separating Hyperplanes.....494
Supporting Hyperplanes.....494
Characterizations for Convex Subsets of E_n:
Separation Theorem.....495
Support Theorem.....496
Representation Theorem.....496

Convex Functions of n=1 Variable
Characterization: Case When Second Derivative Exists....497
Properties....498
Convex Functions of n>1 Variables....499
Sections of a Convex Subset of E_n....500
Restrictions of Convex Functions....502
Jensen's Inequality....502
Concave Functions....503
Arithmetic-Geometric-Harmonic Mean Inequality....503
Hints and Answers....504
References....504

19. Max-Min Problems....507
Statement of Problem
Relative vs. Global Max-Min....509
Critical Points....509
Unconstrained Max-Min: n=1 Variable....510
Unconstrained Max-Min: n>1 Variables....510
Constrained Max-Min....511
Rationale of Lagrange Multipliers....514
Lagrange Function....514
Locating Critical Points....515
Testing Critical Points....516
Limitations....518
Generalizations: Linear and Non-Linear Programming....520
Hints and Answers....521
References....522

20. Some Basic Inequalities....524
Cauchy Inequality:
Finite Series Version....527
Infinite Series Version....530
Complex Series Version....531
Riemann-Stieltjes Integral Version....532
p-q Inequality....534
Hölder Inequality:
Finite Series Version....535
Infinite Series Version....536
Complex Series Version....537
Riemann-Stieltjes Integral Version....537
Triangle Inequality: Finite Series Version....538
Minkowski Inequality:
Finite Series Version....539
Infinite Series Version....540
Riemann-Stieltjes Integral Version....541
c_r Inequality....542
Hints and Answers....543
References....549

Index....551