
Contents

| | | |
|----------|--|-----------|
| I | Topology and Lattices | 1 |
| | Introduction | 3 |
| 1 | Continuous Lattices | |
| | K. Keimel and J. Lawson | 5 |
| | 1-1 Introduction | 5 |
| | 1-1.1 Some terminology | 7 |
| | 1-2 Basics | 7 |
| | 1-3 The equational theory | 12 |
| | 1-4 The Scott topology | 19 |
| | 1-5 Function spaces and Cartesian closed categories | 23 |
| | 1-6 The Lawson topology | 28 |
| | 1-7 Generation by irreducibles and primes | 34 |
| | 1-8 Fixed point theorems and domain equations | 38 |
| | 1-9 Appendix: Galois adjunctions | 45 |
| | 1-10 Exercises | 49 |
| 2 | Frames: Topology Without Points | |
| | A. Pultr and J. Sichler | 55 |
| | 2-1 Introduction | 55 |
| | 2-2 Topological spaces and lattices of open sets: frames | 57 |
| | 2-2.1 Spaces and frames | 57 |
| | 2-2.2 Sobriety | 57 |
| | 2-2.3 Spectrum | 58 |
| | 2-2.4 The spectrum adjunction | 59 |
| | 2-3 Sublocales (generalized subspaces) | 60 |
| | 2-3.1 On the definition | 60 |
| | 2-3.2 Frame congruences | 60 |
| | 2-3.3 Sublocales | 61 |
| | 2-3.4 Open and closed sublocales | 62 |

| | | |
|-------|---|----|
| 2-3.5 | How to construct the congruence generated by a relation | 64 |
| 2-4 | Free frames. Coproduct | 65 |
| 2-4.1 | Free frames: the down-set functor | 65 |
| 2-4.2 | The construction | 66 |
| 2-4.3 | Coproducts of frames | 66 |
| 2-4.4 | The basic elements $\oplus_i a_i$ | 68 |
| 2-4.5 | Products on Loc compared with topological products | 68 |
| 2-5 | Separation axioms | 69 |
| 2-5.1 | Normal, regular, and completely regular frames | 69 |
| 2-5.2 | Subfitness | 70 |
| 2-5.3 | Hausdorff axiom | 71 |
| 2-5.4 | More about regular frames | 71 |
| 2-6 | Compactness and compactification | 72 |
| 2-6.1 | A few concepts | 72 |
| 2-6.2 | Properties | 72 |
| 2-6.3 | Two more counterparts of classical Hausdorff facts | 73 |
| 2-6.4 | A simple but not very satisfactory compactification | 73 |
| 2-7 | Continuous frames; locally compact spaces. | |
| | Hofmann–Lawson duality | 76 |
| 2-7.1 | Continuous lattices | 76 |
| 2-7.2 | Locally compact spaces and continuous frames | 77 |
| 2-7.3 | Adjustment of the spectrum construction | 77 |
| 2-7.4 | Scott topology | 77 |
| 2-8 | Notes on uniform frames | 79 |
| 2-8.1 | Covers and systems of covers | 79 |
| 2-8.2 | Uniformities | 80 |
| 2-8.3 | Completeness and completion | 81 |
| 2-8.4 | An application. Behaviour of paracompact frames | 82 |
| 2-9 | Exercises | 83 |
| 2-10 | Problems | 86 |

II Special Classes of Finite Lattices 89

3 Planar Semimodular Lattices

G. Czédli and G. Grätzer 91

| | | |
|-----|------------------------------|-----|
| 3-1 | Introduction | 91 |
| 3-2 | ◊ Some related results | 92 |
| 3-3 | Planarity and diagrams | 93 |
| 3-4 | Slim lattices, the basics | 97 |
| 3-5 | Construction with forks | 103 |
| 3-6 | Construction with resections | 106 |
| 3-7 | ◊ Rectangular lattices | 108 |
| 3-8 | ◊ A description by matrices | 111 |

| | | |
|----------|--|------------|
| 3-9 | Description by permutations | 114 |
| 3-10 | Variants of the Jordan-Hölder Theorem | 117 |
| 3-11 | Exercises | 121 |
| 4 | Planar Semimodular Lattices: Congruences | |
| | G. Grätzer | 131 |
| 4-1 | Introduction | 131 |
| 4-2 | Congruence structure and N_7 sublattices | 133 |
| 4-2.1 | Congruences in finite lattices | 134 |
| 4-2.2 | Finite semimodular lattices | 136 |
| 4-2.3 | Proof of the Tight N_7 Theorem | 137 |
| 4-3 | Congruence lattices of rectangular lattices | 141 |
| 4-3.1 | Preliminaries | 141 |
| 4-3.2 | The construction and proof | 142 |
| 4-3.3 | The size of L_n | 145 |
| 4-4 | More on tight N_7 -s | 147 |
| 4-5 | The Lower Bound Theorem | 150 |
| 4-6 | Proof of Theorem 4-1.2(ii) | 156 |
| 4-7 | A brief survey of recent results | 157 |
| 4-8 | Exercises | 158 |
| 4-9 | Problems | 164 |
| 5 | Sectionally Complemented Lattices | |
| | G. Grätzer | 167 |
| 5-1 | Introduction | 167 |
| 5-2 | Chopped lattices | 168 |
| 5-2.1 | Basic definitions | 168 |
| 5-2.2 | Compatible vectors | 169 |
| 5-2.3 | From the chopped lattice to the ideal lattice | 170 |
| 5-2.4 | Sectional complementation | 170 |
| 5-3 | The representation theorem | 171 |
| 5-3.1 | Constructing M , congruences | 171 |
| 5-3.2 | L is sectionally complemented | 174 |
| 5-4 | An algorithmic construction of sectional complements | 175 |
| 5-4.1 | A crude algorithm | 175 |
| 5-4.2 | Incompatibilities and failures | 176 |
| 5-4.3 | Failures, cuts, and the algorithm | 177 |
| 5-4.4 | The result | 179 |
| 5-4.5 | Proving the failure lemmas | 179 |
| 5-4.6 | Proving the main result | 180 |
| | V -compatibility | 181 |
| | C -compatibility | 182 |
| | H -compatibility | 182 |
| 5-4.7 | Sectional complement | 183 |

| | | |
|------------|--|------------|
| 5-5 | Convergence | 185 |
| 5-5.1 | Proof of Theorem 5-5.1 | 185 |
| 5-5.2 | Proof of Theorem 5-5.2 | 186 |
| 5-6 | Exercises | 189 |
| 5-7 | Problems | 192 |
| 6 | Combinatorics in Finite Lattices | |
| | J.P.S. Kung | 195 |
| 6-1 | Introduction | 195 |
| 6-2 | Möbius functions | 196 |
| 6-3 | Complements and determinants | 203 |
| 6-4 | Matchings and counting inequalities in lattices | 206 |
| 6-5 | Eulerian functions of groups and closure systems | 210 |
| 6-6 | Characteristic polynomials of geometric lattices | 213 |
| 6-6.1 | Matroids | 213 |
| 6-6.2 | The no-broken-circuit complex | 215 |
| 6-6.3 | A division theorem for characteristic polynomials | 216 |
| 6-7 | Antichains and the Sperner property | 219 |
| 6-8 | Exercises | 223 |
| III | Congruence Lattices of Infinite Lattices, and Beyond | 231 |
| | Introduction | 233 |
| 7 | Schmidt and Pudlák's Approaches to CLP | |
| | F. Wehrung | 235 |
| 7-1 | Introduction | 235 |
| 7-2 | Categorical background | 237 |
| 7-3 | Distributive homomorphisms | 240 |
| 7-3.1 | Algebraic closure operators and congruences of semilattices | 240 |
| 7-3.2 | Weakly distributive homomorphisms and congruences | 241 |
| 7-3.3 | Distributive congruences | 243 |
| 7-3.4 | Schmidt's Theorem | 247 |
| 7-3.5 | Further applications of the Boolean triple construction | 250 |
| 7-4 | From finite to infinite semilattices | 251 |
| 7-4.1 | The Ershov-Pudlák Lemma | 252 |
| 7-4.2 | Directed colimits of finite Boolean semilattices | 253 |
| 7-4.3 | Ladders | 258 |
| 7-4.4 | Pudlák's approach to CLP | 260 |
| 7-4.5 | A very simple unliftable triangle | 263 |
| 7-5 | Representing semilattices of cardinality up to aleph one | 263 |
| 7-5.1 | A one-dimensional amalgamation result for \mathbb{F} -lattices | 264 |

| | | |
|----------|--|------------|
| 7-5.2 | Representing countable distributive semilattices | 265 |
| 7-5.3 | Embedding finite lattices into finite equivalence lattices | 267 |
| 7-5.4 | Representing distributive semilattices with at most \aleph_1 elements | 268 |
| 7-5.5 | Lifting tree-indexed diagrams of semilattices | 275 |
| 7-6 | Congruence amalgamation for infinite lattices | 277 |
| 7-6.1 | Stepwise enlargements of partial lattices | 278 |
| 7-6.2 | Two-dimensional congruence amalgamation theorems for partial lattices | 282 |
| 7-6.3 | One-dimensional congruence amalgamation theorems for partial lattices | 283 |
| 7-6.4 | Three-dimensional congruence amalgamation | 286 |
| 7-7 | Exercises | 287 |
| 7-8 | Problems | 295 |
| 8 | Congruences of Lattices and Ideals of Rings | |
| | F. Wehrung | 297 |
| 8-1 | Introduction | 297 |
| 8-2 | Basic concepts | 298 |
| 8-2.1 | Ideals and congruences of sectionally complemented modular lattices | 298 |
| 8-2.2 | Commutative monoids, refinement monoids, dimension groups | 298 |
| 8-2.3 | Rings and ideals | 300 |
| 8-2.4 | Premodules and modules | 303 |
| 8-3 | Von Neumann regular rings | 305 |
| 8-3.1 | Basic properties of regular rings | 306 |
| 8-3.2 | Principal right ideals in regular rings | 308 |
| 8-3.3 | Neutral ideals of right ideal lattices of regular rings | 312 |
| 8-3.4 | Ideal lattices of ideals and of corner rings | 314 |
| 8-4 | Representing distributive semilattices by regular rings | 315 |
| 8-4.1 | Bergman and Růžička's representation results by locally matricial algebras | 316 |
| 8-4.2 | Nonstable K-theory of rings | 317 |
| 8-4.3 | Sending the nonstable K-theory to the ideal lattice | 319 |
| 8-4.4 | Semilattices with at most \aleph_1 elements | 321 |
| 8-4.5 | Lifting arrows of countable distributive semilattices | 323 |
| 8-5 | Exercises | 324 |
| 8-6 | Problems | 334 |
| 9 | Liftable and Unliftable Diagrams | |
| | F. Wehrung | 337 |
| 9-1 | Introduction | 337 |
| 9-2 | Uniform refinement properties | 337 |

| | | |
|-----------|--|------------|
| 9-2.1 | V-distances of fixed finite type | 338 |
| 9-2.2 | Four uniform refinement properties | 341 |
| 9-2.3 | Semilattices failing URP-like statements | 346 |
| 9-2.4 | Congruence lattices of congruence-permutable algebras | 347 |
| 9-2.5 | A uniform refinement property for K_0 of a regular ring | 348 |
| 9-3 | Non-representable semilattices | 350 |
| 9-3.1 | The Ploščica-Tůma construction | 351 |
| 9-3.2 | Free trees, evaporation, erosion | 358 |
| 9-3.3 | A uniform refinement property for congruence semilattices of all lattices | 361 |
| 9-3.4 | Further non-representation results for distributive semilattices | 362 |
| 9-3.5 | A representation result for all distributive semilattices | 365 |
| 9-4 | Critical points | 366 |
| 9-4.1 | Dual topological spaces | 366 |
| 9-4.2 | Countable critical points | 368 |
| 9-4.3 | More critical points | 370 |
| 9-4.4 | The possible values of critical points | 374 |
| 9-5 | Further topics | 375 |
| 9-5.1 | Congruence-lifting small diagrams of finite Boolean semilattices | 375 |
| 9-5.2 | Congruence-lifting poset-indexed diagrams of semilattices | 378 |
| 9-5.3 | Lifting diagrams of semilattices by diagrams of intervals in subgroup lattices of groups | 379 |
| 9-5.4 | Congruence m -permutable, congruence-preserving extensions | 384 |
| 9-6 | Exercises | 385 |
| 9-7 | Problems | 390 |
| 10 | Two More Topics on Congruence Lattices of Lattices | |
| | G. Grätzer | 393 |
| 10-1 | Introducing complete congruence lattices | 393 |
| 10-1.1 | Four problems: 1945–1983 | 394 |
| 10-1.2 | Four solutions: 1960–1988 | 394 |
| 10-1.3 | Related structures of algebras | 396 |
| 10-1.4 | Related structures of lattices | 396 |
| 10-1.5 | Related structures of complete lattices | 397 |
| 10-2 | The Representation Theorem for Complete Lattices | 397 |
| 10-2.1 | Preliminary steps | 398 |
| 10-2.2 | The lattice M_X | 401 |
| 10-2.3 | The complete lattice K | 403 |
| 10-3 | The Independence Theorem for Complete Lattices | 404 |
| 10-4 | An application to infinitary algebras | 406 |

| | | |
|--------|--|-----|
| 10-4.1 | The construction of the algebra \mathfrak{A} | 406 |
| 10-4.2 | A proof of the Independence Theorem | 408 |
| 10-5 | Complete-simple distributive lattices | 411 |
| 10-5.1 | Why ICSD lattices? | 412 |
| 10-5.2 | The $D^{(2)}$ construction | 412 |
| 10-5.3 | The second construction | 414 |
| 10-5.4 | Support systems | 417 |
| 10-5.5 | Multigluing | 418 |
| 10-5.6 | The proof of the Existence Theorem for ICSD Lattices | 420 |
| 10-6 | The order of principal congruences | 421 |
| 10-6.1 | Principal congruences | 421 |
| 10-6.2 | The result | 423 |
| 10-6.3 | The construction | 423 |
| | The lattice F | 423 |
| | The lattice K | 424 |
| 10-6.4 | The proof | 427 |
| | Preliminaries | 427 |
| | The congruences of S | 427 |
| | The congruences of K | 428 |
| 10-7 | Exercises | 430 |
| 10-8 | Problems | 433 |

| | |
|---------------------------|------------|
| Bibliography | 437 |
| Corrections to LTF | 460 |
| Index | 461 |