

This text presents topos theory as it has developed from the study of sheaves. Sheaves arose in geometry as coefficients for cohomology and as descriptions of the functions appropriate to various kinds of manifolds (algebraic, analytic, etc.). Sheaves also appear in logic as carriers for models of set theory as well as for the semantics of other types of logic. Grothendieck introduced a topos as a category of sheaves for algebraic geometry. Subsequently, Lawvere and Tierney obtained elementary axioms for such (more general) categories.

This introduction to topos theory begins with a number of illustrative examples that explain the origin of these ideas and then describes the sheafification process and the properties of an elementary topos. The applications to axiomatic set theory and the use in forcing (the Independence of the Continuum Hypothesis and of the Axiom of Choice) are then described. Geometric morphisms—like continuous maps of spaces—and the construction of classifying topoi, for example those related to local rings and simplicial sets, next appear, followed by the use of locales (pointless spaces) and the construction of topoi related to geometric languages and logic. This is the first text to address all of these varied aspects of topos theory at the graduate student level.

ISBN 0-387-97710-4



9 780387 977102

www.springer-ny.com

Contents

Preface	vii
Prologue	1
Categorical Preliminaries	10
I. Categories of Functors	24
1. The Categories at Issue	24
2. Pullbacks	29
3. Characteristic Functions of Subobjects	31
4. Typical Subobject Classifiers	35
5. Colimits	39
6. Exponentials	44
7. Propositional Calculus	48
8. Heyting Algebras	50
9. Quantifiers as Adjoints	57
Exercises	62
II. Sheaves of Sets	64
1. Sheaves	65
2. Sieves and Sheaves	69
3. Sheaves and Manifolds	73
4. Bundles	79
5. Sheaves and Cross-Sections	83
6. Sheaves as Étale Spaces	88
7. Sheaves with Algebraic Structure	95
8. Sheaves are Typical	97
9. Inverse Image Sheaf	99
Exercises	103
III. Grothendieck Topologies and Sheaves	106
1. Generalized Neighborhoods	106
2. Grothendieck Topologies	109
3. The Zariski Site	116
4. Sheaves on a Site	121

5. The Associated Sheaf Functor	128
6. First Properties of the Category of Sheaves	134
7. Subobject Classifiers for Sites	140
8. Subsheaves	145
9. Continuous Group Actions	150
Exercises	155
IV. First Properties of Elementary Topoi	161
1. Definition of a Topos	161
2. The Construction of Exponentials	167
3. Direct Image	171
4. Monads and Beck's Theorem	176
5. The Construction of Colimits	180
6. Factorization and Images	184
7. The Slice Category as a Topos	190
8. Lattice and Heyting Algebra Objects in a Topos	198
9. The Beck-Chevalley Condition	204
10. Injective Objects	210
Exercises	213
V. Basic Constructions of Topoi	218
1. Lawvere-Tierney Topologies	219
2. Sheaves	223
3. The Associated Sheaf Functor	227
4. Lawvere-Tierney Subsumes Grothendieck	233
5. Internal Versus External	235
6. Group Actions	237
7. Category Actions	240
8. The Topos of Coalgebras	247
9. The Filter-Quotient Construction	256
Exercises	263
VI. Topoi and Logic	267
1. The Topos of Sets	268
2. The Cohen Topos	277
3. The Preservation of Cardinal Inequalities	284
4. The Axiom of Choice	291
5. The Mitchell-Bénabou Language	296
6. Kripke-Joyal Semantics	302
7. Sheaf Semantics	315
8. Real Numbers in a Topos	318
9. Brouwer's Theorem: All Functions are Continuous	324
10. Topos-Theoretic and Set-Theoretic Foundations	331
Exercises	343

VII. Geometric Morphisms	347
1. Geometric Morphisms and Basic Examples	348
2. Tensor Products	353
3. Group Actions	361
4. Embeddings and Surjections	366
5. Points	378
6. Filtering Functors	384
7. Morphisms into Grothendieck Topoi	390
8. Filtering Functors into a Topos	394
9. Geometric Morphisms as Filtering Functors	399
10. Morphisms Between Sites	407
Exercises	414
VIII. Classifying Topoi	419
1. Classifying Spaces in Topology	420
2. Torsors	423
3. Classifying Topoi	432
4. The Object Classifier	434
5. The Classifying Topos for Rings	437
6. The Zariski Topos Classifies Local Rings	445
7. Simplicial Sets	450
8. Simplicial Sets Classify Linear Orders	455
Exercises	466
IX. Localic Topoi	470
1. Locales	471
2. Points and Sober Spaces	473
3. Spaces from Locales	475
4. Embeddings and Surjections of Locales	480
5. Localic Topoi	487
6. Open Geometric Morphisms	491
7. Open Maps of Locales	500
8. Open Maps and Sites	506
9. The Diaconescu Cover and Barr's Theorem	511
10. The Stone Space of a Complete Boolean Algebra	514
11. Deligne's Theorem	519
Exercises	521
X. Geometric Logic and Classifying Topoi	526
1. First-Order Theories	527
2. Models in Topoi	530
3. Geometric Theories	533
4. Categories of Definable Objects	539

5. Syntactic Sites	553
6. The Classifying Topos of a Geometric Theory	559
7. Universal Models	566
Exercises	569
Appendix: Sites for Topoi	572
1. Exactness Conditions	572
2. Construction of Coequalizers	575
3. The Construction of Sites	578
4. Some Consequences of Giraud's Theorem	587
Epilogue	596
Bibliography	603
Index of Notation	613
Index	617