

An Introduction to Differential Manifolds

Jacques LAFONTAINE

This book is an introduction to differential manifolds. It gives solid preliminaries for more advanced topics: Riemannian manifolds, differential topology, Lie theory. It presupposes little background: the reader is only expected to master basic differential calculus, and a little point-set topology. The book covers the main topics of differential geometry: manifolds, tangent space, vector fields, differential forms, Lie groups, and a few more sophisticated topics such as de Rham cohomology, degree theory and the Gauss-Bonnet theorem for surfaces.

Its ambition is to give solid foundations. In particular, the introduction of "abstract" notions such as manifolds or differential forms is motivated via questions and examples from mathematics or theoretical physics. More than 150 exercises, some of them easy and classical, some others more sophisticated, will help the beginner as well as the more expert reader. Solutions are provided for most of them.

The book should be of interest to various readers: undergraduate and graduate students for a first contact to differential manifolds, mathematicians from other fields and physicists who wish to acquire some feeling about this beautiful theory.

The original French text *Introduction aux variétés différentielles* has been a best-seller in its category in France for many years.

Jacques Lafontaine was successively assistant Professor at Paris Diderot University and Professor at the University of Montpellier, where he is presently emeritus. His main research interests are Riemannian and pseudo-Riemannian geometry, including some aspects of mathematical relativity. Besides his personal research articles, he was involved in several textbooks and research monographs.

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