



Photo by Patrick Studdard.

Professor Bard has provided a valuable service by carefully explaining everything an undergraduate student of mathematics, or a teacher of these topics, needs to get started with Sage quickly and easily. It will also be useful for any student or teacher of another STEM discipline. There is an excellent mix of the most frequently used commands, along with warnings about common pitfalls or caveats. I highly recommend it for anyone new to Sage, or who desires an overview of the system's impressive capabilities.

—Robert A. Beezer, University of Puget Sound

This book is a sort of “Missing Manual” that explains how Sage can be used in a range of standard mathematics courses, instead of targeting specialists like much existing Sage documentation. The depth of content is very impressive, and describes—in a single coherent narrative—how to successfully use Sage for a wide swath of undergraduate applied topics.

—William Stein, University of Washington, Seattle

As the open-source and free competitor to expensive software like Maple™, Mathematica®, Magma, and MATLAB®, Sage offers anyone with access to a web browser the ability to use cutting-edge mathematical software and display his or her results for others, often with stunning graphics. This book is a gentle introduction to Sage for undergraduate students toward the end of Calculus II (single-variable integral calculus) or higher-level course work such as Multivariate Calculus, Differential Equations, Linear Algebra, or Math Modeling.

The book assumes no background in computer science, but the reader who finishes the book will have learned about half of a first semester Computer Science I course, including large parts of the Python programming language. The audience of the book is not only math majors, but also physics, engineering, finance, statistics, chemistry, and computer science majors.

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