Can a Christian escape from a lion? How quickly can a rumour spread? Can you fool an airline into accepting oversize baggage? Recreational mathematics is full of frivolous questions in which the mathematician's art can be brought to bear. But play often has a purpose, whether it's bear cubs in mock fights, or war games. In mathematics, it can sharpen skills, or provide amusement, or simply surprise, and collections of problems have been the stock-in-trade of mathematicians for centuries. Two of the twentieth century's greatest players of problem posing and solving, Erdös and Littlewood, are the inspiration for this collection, which is designed to be sipped from, rather than consumed, in one sitting. The questions themselves range in difficulty: the most challenging offer a glimpse of deep results that engage mathematicians today; even the easiest are capable of prompting readers to think about mathematics. All come with solutions, many with hints, and most with illustrations. Whether you are an expert, or a beginner, or an amateur, this book will delight for a lifetime.

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