

The striking theorems showcased in this book are among the most profound results of twentieth-century analysis. The authors' original approach combines rigorous mathematical proofs with commentary on the underlying ideas to provide a rich insight into these landmarks in mathematics. Results ranging from the proof of Littlewood's conjecture to the Banach–Tarski paradox have been selected for their mathematical beauty as well as their educative value and historical role. Placing each theorem in historical perspective, the authors paint a coherent picture of modern analysis and its development, whilst maintaining mathematical rigour with the provision of complete proofs, alternative proofs, worked examples, and more than 150 exercises and solution hints.

This edition extends the original French edition of 2009 with a new chapter on partitions, including the Hardy–Ramanujan theorem, and a significant expansion of the existing chapter on the corona problem.

"There is a fine balance of methods of 'hard' and 'soft' analysis, which are often beautifully combined. The style is vivid and even entertaining at times ... [The review] can only give a glimpse of what this beautifully written, rich and almost unique monograph contains."

*Zentralblatt MATH*

"[makes] you want to read and learn ... those who love analysis and beautiful demonstrations will read and re-read with pleasure and profit."

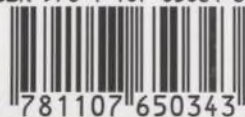
*Gazette des mathématiciens*

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