

Springer Monographs in Mathematics

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Local Homotopy Theory

This monograph on the homotopy theory of topologized diagrams of spaces and spectra gives an expert account of a subject at the foundation of motivic homotopy theory and the theory of topological modular forms in stable homotopy theory.

Beginning with an introduction to the homotopy theory of simplicial sets and topos theory, the book covers core topics such as the unstable homotopy theory of simplicial presheaves and sheaves, localized theories, cocycles, descent theory, non-abelian cohomology, stacks, and local stable homotopy theory. A detailed treatment of the formalism of the subject is interwoven with explanations of the motivation, development, and nuances of ideas and results. The coherence of the abstract theory is elucidated through the use of widely applicable tools, such as Barr's theorem on Boolean localization, model structures on the category of simplicial presheaves on a site, and cocycle categories. A wealth of concrete examples convey the vitality and importance of the subject in topology, number theory, algebraic geometry, and algebraic K -theory.

Assuming basic knowledge of algebraic geometry and homotopy theory, *Local Homotopy Theory* will appeal to researchers and advanced graduate students seeking to understand and advance the applications of homotopy theory in multiple areas of mathematics and the mathematical sciences.

Mathematics
ISSN 1439-7382

ISBN 978-1-4939-2299-4



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