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Infinity Properads and Infinity Wheeled Properads

The topic of this book sits at the interface of the theory of higher categories (in the guise of $(\infty, 1)$ -categories) and the theory of properads. Properads are devices more general than operads, and enable one to encode bialgebraic, rather than just (co) algebraic, structures.

The text extends both the Joyal-Lurie approach to higher categories and the Cisinski-Moerdijk-Weiss approach to higher operads, and provides a foundation for a broad study of the homotopy theory of properads. This work also serves as a complete guide to the generalised graphs which are pervasive in the study of operads and properads. A preliminary list of potential applications and extensions comprises the final chapter.

Infinity Properads and Infinity Wheeled Properads is written for mathematicians in the fields of topology, algebra, category theory, and related areas. It is written roughly at the second year graduate level, and assumes a basic knowledge of category theory.

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