A Comprehensive Course in Analysis by Poincaré Prize winner Barry Simon is a five-volume set that can serve as a graduate-level analysis textbook with a lot of additional *bonus* information, including hundreds of problems and numerous notes that extend the text and provide important historical background. Depth and breadth of exposition make this set a valuable reference source for almost all areas of classical analysis.

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Part 2A is devoted to basic complex analysis. It interweaves three analytic threads associated with Cauchy, Riemann, and Weierstrass, respectively. Cauchy's view focuses on the differential and integral calculus of functions of a complex variable, with the key topics being the Cauchy integral formula and contour integration. For Riemann, the geometry of the complex plane is central, with key topics being fractional linear transformations and conformal mapping. For Weierstrass, the power series is king, with key topics being spaces of analytic functions, the product formulas of Weierstrass and Hadamard, and the Weierstrass theory of elliptic functions. Subjects in this volume that are often missing in other texts include the Cauchy integral theorem when the contour is the boundary of a Jordan region, continued fractions, two proofs of the big Picard theorem, the uniformization theorem, Ahlfors's function, the sheaf of analytic germs, and Jacobi, as well as Weierstrass, elliptic functions.

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