

# Directional Statistics

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Observations which are directions, axes or rotations occur in many sciences, including astronomy, biology, earth sciences, image analysis, and medicine. To analyse such data it is necessary to use the techniques of directional statistics, in which the special structure of circles, spheres and rotation groups is taken into account. This book gives a unified and comprehensive account of directional statistics, presenting both the underlying statistical theory and the practical methodology.

The book is divided into three parts. The first part concentrates on statistics on the circle. Topics covered include tests of uniformity, tests of goodness-of-fit, inference on von Mises distributions and non-parametric methods. The second part considers statistics on spheres of arbitrary dimension, and includes a detailed account of inference on the main distributions on spheres. Recent material on correlation, regression, time series, robust techniques, bootstrap methods, density estimation and curve fitting is presented. The third part considers statistics on more general sample spaces, in particular rotation groups, Stiefel manifolds, Grassmann manifolds and complex projective spaces. Shape analysis is considered from the perspective of directional statistics.

This text will be invaluable not only to researchers in probability and statistics interested in the latest developments in directional statistics, but also to practitioners and researchers in many scientific fields, including astronomy, biology, computer vision, earth sciences and image analysis.

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