

Introduction to Modern Finsler Geometry

This comprehensive book is an introduction to the basics of Finsler geometry with recent developments in its area. It includes local geometry as well as global geometry of Finsler manifolds.

In Part I, the authors discuss differential manifolds, Finsler metrics, the Chern connection, Riemannian and non-Riemannian quantities. Part II is written for readers who would like to further their studies in Finsler geometry. It covers projective transformations, comparison theorems, fundamental group, minimal immersions, harmonic maps, Einstein metrics, conformal transformations, amongst other related topics. The authors made great efforts to ensure that the contents are accessible to senior undergraduate students, graduate students, mathematicians and scientists.

Higher Education Press

www.hep.com.cn

World Scientific

www.worldscientific.com

9726 hc

ISBN 978-981-4704-90-8



9 789814 704908

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