This book provides a broad introduction to the subject of dynamical systems, suitable for a one- or two-semester graduate course. In the first chapter, the authors introduce over a dozen examples, and then use these examples throughout the book to motivate and clarify the development of the theory. Topics include topological dynamics, symbolic dynamics, ergodic theory, hyperbolic dynamics, one-dimensional dynamics, complex dynamics, and measure-theoretic entropy. The authors top off the presentation with some beautiful and remarkable applications of dynamical systems to such areas as number theory, data storage, and Internet search engines.

This book grew out of lecture notes from the graduate dynamical systems course at the University of Maryland, College Park, and reflects not only the tastes of the authors, but also to some extent the collective opinion of the Dynamics Group at the University of Maryland, which includes experts in virtually every major area of dynamical systems.

"exceptionally well written... You should consider adopting this wonderful text for your next graduate course on the pure mathematics of the modern theory of dynamical systems." *SIAM Review*

"... despite the breadth, one finds here major results rigorously treated and substantial applications. By itself, the clean, accessible exposition of the amazing Sharkovsky theorem would justify the acquisition of this book.... Highly recommended." *Choice*

Michael Brin is Professor Emeritus of mathematics at the University of Maryland. He is the author of over 30 papers, three of which appeared in the *Annals of Mathematics*, and he has lectured at conferences and universities around the world. His main research areas are dynamical systems and Riemannian geometry. In 2008, he established the Michael Brin Prize in Dynamical Systems.

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Contents

Introduction *page* xi

1 Examples and basic concepts 1

1.1 The notion of a dynamical system 1
1.2 Circle rotations 3
1.3 Expanding endomorphisms of the circle 5
1.4 Shifts and subshifts 7
1.5 Quadratic maps 9
1.6 The Gauss transformation 11
1.7 Hyperbolic toral automorphisms 14
1.8 The horseshoe 16
1.9 The solenoid 17
1.10 Flows and differential equations 19
1.11 Suspension and cross-section 22
1.12 Chaos and Lyapunov exponents 23
1.13 Attractors 25

2 Topological dynamics 29

2.1 Limit sets and recurrence 29
2.2 Topological transitivity 32
2.3 Topological mixing 34
2.4 Expansiveness 36
2.5 Topological entropy 37
2.6 Topological entropy for some examples 42
2.7 Equicontinuity, distality and proximality 46
2.8 Topological recurrence and Ramsey Theory 49

3 Symbolic dynamics 55

3.1 Subshifts and codes 56
3.2 Subshifts of finite type 57

3.3 The Perron–Frobenius Theorem 58
3.4 Topological entropy and the zeta function of an SFT 61
3.5 Strong shift equivalence and shift equivalence 63
3.6 Substitutions 64
3.7 Sofic shifts 67
3.8 Data storage 68

4 Ergodic theory 70
4.1 Measure theory preliminaries 70
4.2 Recurrence 73
4.3 Ergodicity and mixing 74
4.4 Examples 78
4.5 Ergodic theorems 81
4.6 Invariant measures for continuous maps 86
4.7 Unique ergodicity and Weyl's Theorem 88
4.8 The Gauss transformation revisited 92
4.9 Discrete spectrum 96
4.10 Weak mixing 98
4.11 Applications of measure-theoretic recurrence to number theory 102
4.12 Internet search 105

5 Hyperbolic dynamics 108
5.1 Expanding endomorphisms revisited 109
5.2 Hyperbolic sets 110
5.3 ϵ-orbits 112
5.4 Invariant cones 116
5.5 Stability of hyperbolic sets 119
5.6 Stable and unstable manifolds 120
5.7 Inclination Lemma 124
5.8 Horseshoes and transverse homoclinic points 126
5.9 Local product structure and locally maximal hyperbolic sets 130
5.10 Anosov diffeomorphisms 132
5.11 Axiom A and structural stability 135
5.12 Markov partitions 137
5.13 Appendix: Differentiable manifolds 140

6 Ergodicity of Anosov diffeomorphisms 144
6.1 Hölder continuity of the stable and unstable distributions 144

6.2 Absolute continuity of the stable and unstable foliations 147
6.3 Proof of ergodicity 154

7 Low dimensional dynamics 157
7.1 Circle homeomorphisms 157
7.2 Circle diffeomorphisms 164
7.3 The Sharkovsky Theorem 166
7.4 Combinatorial theory of piecewise-monotone mappings 174
7.5 The Schwarzian derivative 182
7.6 Real quadratic maps 186
7.7 Bifurcations of periodic points 187
7.8 The Feigenbaum phenomenon 193

8 Complex dynamics 196
8.1 Complex analysis on the Riemann sphere 196
8.2 Examples 199
8.3 Normal families 202
8.4 Periodic points 203
8.5 The Julia set 205
8.6 The Mandelbrot set 211

9 Measure theoretic entropy 213
9.1 Entropy of a partition 213
9.2 Conditional entropy 216
9.3 Entropy of a measure-preserving transformation 218
9.4 Examples of entropy calculation 223
9.5 Variational principle 226

Bibliography 230
Index 237