

Florian Cajori

A HISTORY OF MATHEMATICAL NOTATIONS

Two Volumes Bound As One

This classic study remains one of the most comprehensive and illuminating histories of mathematical notation ever written. The principal intent of the book is to provide, beyond an interesting array of historic facts, useful perspectives on how the successes and failures of past mathematical symbols might relate to the notational problems of contemporary mathematics. Originally published in a two-volume edition, this monumental work is presented here in one volume.

Throughout the study, Professor Cajori, a distinguished mathematician, author and educator, notes the first appearance of a symbol and its origin, the competition it encountered, its spread among writers in different countries, its rise to popularity, its eventual decline or ultimate survival. His coverage of obsolete notations—and what we can learn from them—is as comprehensive as those which have survived and still enjoy favor.

Volume I introduces the subject and examines numeral symbols and combinations of symbols, and symbols in arithmetic, algebra and geometry. Volume II contains an advanced topical survey of symbols in arithmetic, algebra, modern analysis and geometry, and an exploration of the teachings of history. Over one hundred figures and diagrams illustrate the materials presented. Florian Cajori, author of a number of classic textbooks in mathematics and physics, taught at Tulane University, Colorado College and the University of California.

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