

1	A Garden of Integers	1
1.1	Figurate numbers	1
1.2	Sums of squares, triangular numbers, and cubes	6
1.3	There are infinitely many primes	9
1.4	Fibonacci numbers	12
1.5	Fermat's theorem	15
1.6	Wilson's theorem	16
1.7	Perfect numbers	16
1.8	Challenges	17
2	Distinguished Numbers	19
2.1	The irrationality of $\sqrt{2}$	20
2.2	The irrationality of $\sqrt{k}$ for non-square k	21
2.3	The golden ratio	22
2.4	π and the circle	25
2.5	The irrationality of π	27
2.6	The Comte de Buffon and his needle	28
2.7	e as a limit	29
2.8	An infinite series for e	32
2.9	The irrationality of e	32
2.10	Steiner's problem on the number e	33
2.11	The Euler-Mascheroni constant	34
2.12	Exponents, rational and irrational	35
2.13	Challenges	36
3	Points in the Plane	39
3.1	Pick's theorem	39
3.2	Circles and sums of two squares	41
3.3	The Sylvester-Gallai theorem	43

3.4	Bisecting a set of 100,000 points	44
3.5	Pigeons and pigeonholes	45
3.6	Assigning numbers to points in the plane	47
3.7	Challenges	48
4	The Polygonal Playground	51
4.1	Polygonal combinatorics	51
4.2	Drawing an n -gon with given side lengths	54
4.3	The theorems of Maekawa and Kawasaki	55
4.4	Squaring polygons	58
4.5	The stars of the polygonal playground	59
4.6	Guards in art galleries	62
4.7	Triangulation of convex polygons	63
4.8	Cycloids, cyclogons, and polygonal cycloids	66
4.9	Challenges	69
5	A Treasury of Triangle Theorems	71
5.1	The Pythagorean theorem	71
5.2	Pythagorean relatives	73
5.3	The inradius of a right triangle	75
5.4	Pappus' generalization of the Pythagorean theorem	77
5.5	The incircle and Heron's formula	78
5.6	The circumcircle and Euler's triangle inequality	80
5.7	The orthic triangle	81
5.8	The Erdős-Mordell inequality	82
5.9	The Steiner-Lehmus theorem	84
5.10	The medians of a triangle	85
5.11	Are most triangles obtuse?	87
5.12	Challenges	88
6	The Enchantment of the Equilateral Triangle	91
6.1	Pythagorean-like theorems	91
6.2	The Fermat point of a triangle	94
6.3	Viviani's theorem	96
6.4	A triangular tiling of the plane and Weitzenböck's inequality	96
6.5	Napoleon's theorem	99
6.6	Morley's miracle	100
6.7	Van Schooten's theorem	102
6.8	The equilateral triangle and the golden ratio	103
6.9	Challenges	104

7	The Quadrilaterals' Corner	107
7.1	Midpoints in quadrilaterals	107
7.2	Cyclic quadrilaterals	109
7.3	Quadrilateral equalities and inequalities	111
7.4	Tangential and bicentric quadrilaterals	115
7.5	Anne's and Newton's theorems	116
7.6	Pythagoras with a parallelogram and equilateral triangles . .	118
7.7	Challenges	119
8	Squares Everywhere	121
8.1	One-square theorems	121
8.2	Two-square theorems	123
8.3	Three-square theorems	128
8.4	Four and more squares	131
8.5	Squares in recreational mathematics	133
8.6	Challenges	135
9	Curves Ahead	137
9.1	Squarable lunes	137
9.2	The amazing Archimedean spiral	144
9.3	The quadratrix of Hippias	146
9.4	The shoemaker's knife and the salt cellar	147
9.5	The Quetelet and Dandelin approach to conics	149
9.6	Archimedes triangles	151
9.7	Helices	154
9.8	Challenges	156
10	Adventures in Tiling and Coloring	159
10.1	Plane tilings and tessellations	160
10.2	Tiling with triangles and quadrilaterals	164
10.3	Infinitely many proofs of the Pythagorean theorem	167
10.4	The leaping frog	170
10.5	The seven friezes	172
10.6	Colorful proofs	175
10.7	Dodecahedra and Hamiltonian circuits	185
10.8	Challenges	186
11	Geometry in Three Dimensions	191
11.1	The Pythagorean theorem in three dimensions	192
11.2	Partitioning space with planes	193

11.3	Corresponding triangles on three lines	195
11.4	An angle-trisecting cone	196
11.5	The intersection of three spheres	197
11.6	The fourth circle	199
11.7	The area of a spherical triangle	199
11.8	Euler's polyhedral formula	200
11.9	Faces and vertices in convex polyhedra	202
11.10	Why some types of faces repeat in polyhedra	204
11.11	Euler and Descartes à la Pólya	205
11.12	Squaring squares and cubing cubes	206
11.13	Challenges	208
12	Additional Theorems, Problems, and Proofs	209
12.1	Denumerable and nondenumerable sets	209
12.2	The Cantor-Schröder-Bernstein theorem	211
12.3	The Cauchy-Schwarz inequality	212
12.4	The arithmetic mean-geometric mean inequality	214
12.5	Two pearls of origami	216
12.6	How to draw a straight line	218
12.7	Some gems in functional equations	220
12.8	Functional inequalities	227
12.9	Euler's series for $\pi^2/6$	230
12.10	The Wallis product	233
12.11	Stirling's approximation of $n!$	234
12.12	Challenges	236
	Solutions to the Challenges	239
	Chapter 1	239
	Chapter 2	241
	Chapter 3	245
	Chapter 4	247
	Chapter 5	250
	Chapter 6	255
	Chapter 7	258
	Chapter 8	261
	Chapter 9	263
	Chapter 10	265

Chapter 11	270
Chapter 12	272
References	275
Index	289
About the Authors	295