

## Studies in the Development of Modern Mathematics

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### FOURIER SERIES AND WAVELETS

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This comprehensive monograph presents the history and achievements of one of the most important figures in modern mathematics, covering the work of Fourier from his first memoir on the Analytical Theory of Heat to the latest developments in wavelet theory. The work is divided into two parts: the first deals with Fourier series in the classical sense, decomposition of a function into harmonic components, while the second part expounds the modern theory of wavelets – the most recent tool in pure and applied harmonic analysis.

Originally, Fourier series were used to describe and compute the functions which occur in heat diffusion and equilibrium, but they soon led to the development of new theories by Fourier's followers, and some of these original papers are considered here. The diverse applications of classical Fourier series and wavelets, covering such areas as theoretical physics, image analysis and telecommunications, means that this book will be of interest to mathematicians, engineers and physicists alike. The second part of the book is a self-contained exposition, and may serve as a reference on wavelets.

#### About the Authors

Professor Kahane and Dr Lemarié-Rieusset are currently involved in research at the Department of Mathematics, Université de Paris-Sud in France. Dr Lemarié-Rieusset has long been interested in the theory of wavelets and published the first paper on the subject with Yves Meyer in 1986.

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