

AN UPDATE OF ONE OF THE MOST TRUSTED BOOKS ON CONSTRUCTING AND ANALYZING ACTUARIAL MODELS

Written by three renowned authorities in the actuarial field, *Loss Models, Third Edition* upholds the reputation for excellence that has made this book required reading for the Society of Actuaries (SOA) and Casualty Actuarial Society (CAS) qualification examinations. This update serves as a complete presentation of statistical methods for measuring risk and building models to measure loss in real-world events.

This book maintains an approach to modeling and forecasting that utilizes tools related to risk theory, loss distributions, and survival models. Random variables, basic distributional quantities, the recursive method, and techniques for classifying and creating distributions are also discussed. Both parametric and non-parametric estimation methods are thoroughly covered along with advice for choosing an appropriate model. Features of the Third Edition include:

- Extended discussion of risk management and risk measures, including Tail-Value-at-Risk (TVaR)
- New sections on extreme value distributions and their estimation
- Inclusion of homogeneous, nonhomogeneous, and mixed Poisson processes
- Expanded coverage of copula models and their estimation
- Additional treatment of methods for constructing confidence regions when there is more than one parameter

The book continues to distinguish itself by providing over 400 exercises that have appeared on previous SOA and CAS examinations. Intriguing examples from the fields of insurance and business are discussed throughout, and all data sets are available on the book's FTP site, along with programs that assist with conducting loss model analysis.

Loss Models, Third Edition is an essential resource for students and aspiring actuaries who are preparing to take the SOA and CAS preliminary examinations. It is also a must-have reference for professional actuaries, graduate students in the actuarial field, and anyone who works with loss and risk models in their everyday work.

STUART A. KLUGMAN, PhD, is Principal Financial Group Distinguished Professor of Actuarial Science at Drake University. A Fellow of the Society of Actuaries, Dr. Klugman was vice president of the SOA from 2001–2003.

HARRY H. PANJER, PhD, is Professor Emeritus in the Department of Statistics and Actuarial Science at the University of Waterloo, Canada. Past president of both the Canadian Institute of Actuaries and the Society of Actuaries, Dr. Panjer has published numerous articles on risk modeling in the fields of finance and actuarial science.

GORDON E. WILLMOT, PhD, is Munich Re Chair in Insurance and Professor in the Department of Statistics and Actuarial Science at the University of Waterloo, Canada. Dr. Willmot has authored or coauthored over sixty published articles in the areas of risk theory, queueing theory, distribution theory, and stochastic modeling in insurance.

Subscribe to our free Statistics eNewsletter at
wiley.com/enewsletters

Visit wiley.com/statistics

 **WILEY**
wiley.com

ISBN 978-0-470-18781-4



9 780470 187814

CONTENTS

Preface		xvii
PART I INTRODUCTION		
1 Modeling		3
1.1 The model-based approach		3
1.1.1 The modeling process		4
1.1.2 The modeling advantage		5
1.2 Organization of this book		5
2 Random variables		9
2.1 Introduction		9
2.2 Key functions and four models		11
2.2.1 Exercises		19
3 Basic distributional quantities		21
3.1 Moments		21
3.1.1 Exercises		28
3.2 Quantiles		29
3.2.1 Exercises		29
3.3 Generating functions and sums of random variables		30

3.3.1	Exercises	34
3.4	Tails of distributions	34
3.4.1	Classification based on moments	34
3.4.2	Comparison based on limiting tail behavior	35
3.4.3	Classification based on the hazard rate function	36
3.4.4	Classification based on the mean excess loss function	37
3.4.5	Equilibrium distributions and tail behavior	39
3.4.6	Exercises	40
3.5	Measures of Risk	42
3.5.1	Introduction	42
3.5.2	Risk measures and coherence	42
3.5.3	Value-at-Risk	44
3.5.4	Tail-Value-at-Risk	45
3.5.5	Exercises	49
PART II ACTUARIAL MODELS		
4	Characteristics of actuarial models	53
4.1	Introduction	53
4.2	The role of parameters	53
4.2.1	Parametric and scale distributions	54
4.2.2	Parametric distribution families	56
4.2.3	Finite mixture distributions	56
4.2.4	Data-dependent distributions	58
4.2.5	Exercises	59
5	Continuous models	61
5.1	Introduction	61
5.2	Creating new distributions	61
5.2.1	Multiplication by a constant	62
5.2.2	Raising to a power	62
5.2.3	Exponentiation	64
5.2.4	Mixing	64
5.2.5	Frailty models	68
5.2.6	Splicing	69
5.2.7	Exercises	70
5.3	Selected distributions and their relationships	74
5.3.1	Introduction	74
5.3.2	Two parametric families	74
5.3.3	Limiting distributions	74
5.3.4	Exercises	76
5.4	The linear exponential family	77

5.4.1	Exercises	79
5.5	TVaR for continuous distributions	79
5.5.1	Continuous elliptical distributions	80
5.5.2	TVaR for the linear exponential family	82
5.5.3	Exercise	84
5.6	Extreme value distributions	84
5.6.1	Introduction	84
5.6.2	Distribution of the maximum	86
5.6.3	Stability of the maximum of the extreme value distribution	90
5.6.4	The Fisher–Tippett theorem	91
5.6.5	Maximum domain of attraction	93
5.6.6	Generalized Pareto distributions	95
5.6.7	Stability of excesses of the generalized Pareto	96
5.6.8	Limiting distributions of excesses	98
5.6.9	TVaR for extreme value distributions	98
5.6.10	Further reading	100
5.6.11	Exercises	100
6	Discrete distributions and processes	101
6.1	Introduction	101
6.1.1	Exercise	102
6.2	The Poisson distribution	102
6.3	The negative binomial distribution	105
6.4	The binomial distribution	107
6.5	The $(a, b, 0)$ class	108
6.5.1	Exercises	111
6.6	Counting processes	111
6.6.1	Introduction and definitions	111
6.6.2	Poisson processes	114
6.6.3	Processes with contagion	116
6.6.4	Other processes	119
6.6.5	Exercises	120
6.7	Truncation and modification at zero	121
6.7.1	Exercises	126
6.8	Compound frequency models	126
6.8.1	Exercises	132
6.9	Further properties of the compound Poisson class	132
6.9.1	Exercises	137
6.10	Mixed frequency distributions	137
6.10.1	General mixed frequency distribution	137
6.10.2	Mixed Poisson distributions	139
6.10.3	Exercises	144

6.11	Mixed Poisson processes	145
6.11.1	Exercises	149
6.12	Effect of exposure on frequency	151
6.13	An inventory of discrete distributions	152
6.13.1	Exercises	152
6.14	TVaR for discrete distributions	155
6.14.1	TVaR for the discrete linear exponential family	156
6.14.2	Exercises	159
7	Multivariate models	161
7.1	Introduction	161
7.2	Sklar's theorem and copulas	162
7.3	Measures of dependency	163
7.3.1	Spearman's rho	164
7.3.2	Kendall's tau	164
7.4	Tail dependence	165
7.5	Archimedean copulas	166
7.5.1	Exercise	171
7.6	Elliptical copulas	171
7.6.1	Exercise	173
7.7	Extreme value copulas	174
7.7.1	Exercises	176
7.8	Archimax copulas	177
8	Frequency and severity with coverage modifications	179
8.1	Introduction	179
8.2	Deductibles	179
8.2.1	Exercises	184
8.3	The loss elimination ratio and the effect of inflation for ordinary deductibles	185
8.3.1	Exercises	186
8.4	Policy limits	187
8.4.1	Exercises	189
8.5	Coinsurance, deductibles, and limits	189
8.5.1	Exercises	191
8.6	The impact of deductibles on claim frequency	192
8.6.1	Exercises	196
9	Aggregate loss models	199
9.1	Introduction	199
9.1.1	Exercises	202

9.2	Model choices	202
9.2.1	Exercises	203
9.3	The compound model for aggregate claims	203
9.3.1	Exercises	211
9.4	Analytic results	217
9.4.1	Exercises	223
9.5	Computing the aggregate claims distribution	225
9.6	The recursive method	227
9.6.1	Applications to compound frequency models	228
9.6.2	Underflow/overflow problems	230
9.6.3	Numerical stability	230
9.6.4	Continuous severity	231
9.6.5	Constructing arithmetic distributions	231
9.6.6	Exercises	234
9.7	The impact of individual policy modifications on aggregate payments	238
9.7.1	Exercises	241
9.8	Inversion methods	241
9.8.1	Fast Fourier transform	242
9.8.2	Direct numerical inversion	244
9.8.3	Exercise	246
9.9	Calculations with approximate distributions	246
9.9.1	Arithmetic distributions	246
9.9.2	Empirical distributions	249
9.9.3	Piecewise linear cdf	250
9.9.4	Exercises	251
9.10	Comparison of methods	252
9.11	The individual risk model	253
9.11.1	The model	253
9.11.2	Parametric approximation	255
9.11.3	Compound Poisson approximation	256
9.11.4	Exercises	259
9.12	TVaR for aggregate losses	261
9.12.1	TVaR for discrete aggregate loss distributions	262
9.12.2	Aggregate TVaR for some frequency distributions	262
9.12.3	Aggregate TVaR for some severity distributions	264
9.12.4	Summary	267
9.12.5	Exercises	268
10	Discrete-time ruin models	269
10.1	Introduction	269
10.2	Process models for insurance	270
10.2.1	Processes	270

10.2.2	An insurance model	271
10.2.3	Ruin	272
10.3	Discrete, finite-time ruin probabilities	273
10.3.1	The discrete-time process	273
10.3.2	Evaluating the probability of ruin	274
10.3.3	Exercises	276
11	Continuous-time ruin models	277
11.1	Introduction	277
11.1.1	The Poisson process	277
11.1.2	The continuous-time problem	278
11.2	The adjustment coefficient and Lundberg's inequality	279
11.2.1	The adjustment coefficient	279
11.2.2	Lundberg's inequality	283
11.2.3	Exercises	284
11.3	An integrodifferential equation	286
11.3.1	Exercises	290
11.4	The maximum aggregate loss	291
11.4.1	Exercises	294
11.5	Cramer's asymptotic ruin formula and Tijms' approximation	295
11.5.1	Exercises	300
11.6	The Brownian motion risk process	302
11.7	Brownian motion and the probability of ruin	306

PART III CONSTRUCTION OF EMPIRICAL MODELS

12	Review of mathematical statistics	315
12.1	Introduction	315
12.2	Point estimation	316
12.2.1	Introduction	316
12.2.2	Measures of quality	317
12.2.3	Exercises	322
12.3	Interval estimation	324
12.3.1	Exercises	326
12.4	Tests of hypotheses	326
12.4.1	Exercise	330
13	Estimation for complete data	331
13.1	Introduction	331
13.2	The empirical distribution for complete, individual data	335
13.2.1	Exercises	338

13.3	Empirical distributions for grouped data	339
13.3.1	Exercises	341
14	Estimation for modified data	343
14.1	Point estimation	343
14.1.1	Exercises	349
14.2	Means, variances, and interval estimation	351
14.2.1	Exercises	360
14.3	Kernel density models	362
14.3.1	Exercises	365
14.4	Approximations for large data sets	366
14.4.1	Introduction	366
14.4.2	Kaplan–Meier type approximations	367
14.4.3	Exercises	370

PART IV PARAMETRIC STATISTICAL METHODS

15	Parameter estimation	375
15.1	Method of moments and percentile matching	375
15.1.1	Exercises	378
15.2	Maximum likelihood estimation	381
15.2.1	Introduction	381
15.2.2	Complete, individual data	383
15.2.3	Complete, grouped data	384
15.2.4	Truncated or censored data	385
15.2.5	Exercises	388
15.3	Variance and interval estimation	393
15.3.1	Exercises	399
15.4	Non-normal confidence intervals	402
15.4.1	Exercise	404
15.5	Bayesian estimation	404
15.5.1	Definitions and Bayes' theorem	404
15.5.2	Inference and prediction	407
15.5.3	Conjugate prior distributions and the linear exponential family	414
15.5.4	Computational issues	415
15.5.5	Exercises	416
15.6	Estimation for discrete distributions	422
15.6.1	Poisson	422
15.6.2	Negative binomial	425
15.6.3	Binomial	427
15.6.4	The $(a, b, 1)$ class	430

15.6.5	Compound models	434
15.6.6	Effect of exposure on maximum likelihood estimation	435
15.6.7	Exercises	436
16	Model selection	441
16.1	Introduction	441
16.2	Representations of the data and model	442
16.3	Graphical comparison of the density and distribution functions	443
16.3.1	Exercises	447
16.4	Hypothesis tests	448
16.4.1	Kolmogorov–Smirnov test	448
16.4.2	Anderson–Darling test	450
16.4.3	Chi-square goodness-of-fit test	451
16.4.4	Likelihood ratio test	455
16.4.5	Exercises	456
16.5	Selecting a model	459
16.5.1	Introduction	459
16.5.2	Judgment-based approaches	459
16.5.3	Score-based approaches	460
16.5.4	Exercises	467
17	Estimation and model selection for more complex models	473
17.1	Extreme value models	473
17.1.1	Introduction	473
17.1.2	Parameter estimation	474
17.2	Copula models	484
17.2.1	Introduction	484
17.2.2	Maximum likelihood estimation	484
17.2.3	Semiparametric estimation of the copula	486
17.2.4	The role of deductibles	487
17.2.5	Goodness-of-fit testing	489
17.2.6	An example	490
17.2.7	Exercise	491
17.3	Models with covariates	492
17.3.1	Introduction	492
17.3.2	Proportional hazards models	493
17.3.3	The generalized linear and accelerated failure time models	498
17.3.4	Exercises	501

18	Five examples	503
18.1	Introduction	503
18.2	Time to death	503
18.2.1	The data	503
18.2.2	Some calculations	505
18.2.3	Exercise	506
18.3	Time from incidence to report	506
18.3.1	The problem and some data	507
18.3.2	Analysis	507
18.4	Payment amount	508
18.4.1	The data	508
18.4.2	The first model	510
18.4.3	The second model	512
18.5	An aggregate loss example	512
18.6	Another aggregate loss example	516
18.6.1	Distribution for a single policy	516
18.6.2	One hundred policies—excess of loss	516
18.6.3	One hundred policies—aggregate stop-loss	517
18.6.4	Numerical convolutions	519
18.7	Comprehensive exercises	520

PART V ADJUSTED ESTIMATES

19	Interpolation and smoothing	527
19.1	Introduction	527
19.2	Polynomial interpolation and smoothing	529
19.2.1	Exercises	533
19.3	Cubic spline interpolation	533
19.3.1	Construction of cubic splines	534
19.3.2	Exercises	541
19.4	Approximating functions with splines	542
19.4.1	Exercise	545
19.5	Extrapolating with splines	546
19.5.1	Exercise	546
19.6	Smoothing splines	546
19.6.1	Exercise	554
20	Credibility	555
20.1	Introduction	555
20.2	Limited fluctuation credibility theory	557
20.2.1	Full credibility	558
20.2.2	Partial credibility	561

20.2.3	Problems with the approach	564
20.2.4	Notes and References	565
20.2.5	Exercises	565
20.3	Greatest accuracy credibility theory	567
20.3.1	Introduction	567
20.3.2	Conditional distributions and expectation	570
20.3.3	The Bayesian methodology	573
20.3.4	The credibility premium	581
20.3.5	The Buhlmann model	584
20.3.6	The Buhlmann–Straub model	588
20.3.7	Exact credibility	593
20.3.8	Linear versus Bayesian versus no credibility	597
20.3.9	Notes and References	604
20.3.10	Exercises	604
20.4	Empirical Bayes parameter estimation	617
20.4.1	Nonparametric estimation	620
20.4.2	Semiparametric estimation	630
20.4.3	Parametric estimation	632
20.4.4	Notes and References	636
20.4.5	Exercises	637

PART VI SIMULATION

21	Simulation	643
21.1	Basics of simulation	643
21.1.1	The simulation approach	644
21.1.2	Exercises	649
21.2	Examples of simulation in actuarial modeling	649
21.2.1	Aggregate loss calculations	649
21.2.2	Examples of lack of independence or identical distributions	650
21.2.3	Simulation analysis of the two examples	651
21.2.4	Simulating copulas	653
21.2.5	Using simulation to determine risk measures	656
21.2.6	Statistical analyses	656
21.2.7	Exercises	658
21.3	Examples of simulation in finance	660
21.3.1	Investment guarantees	661
21.3.2	Option valuation	662
21.3.3	Exercise	664

Appendix A: An inventory of continuous distributions	665
A.1 Introduction	665
A.2 Transformed beta family	669
A.2.1 Four-parameter distribution	669
A.2.2 Three-parameter distributions	669
A.2.3 Two-parameter distributions	671
A.3 Transformed gamma family	673
A.3.1 Three-parameter distributions	673
A.3.2 Two-parameter distributions	674
A.3.3 One-parameter distributions	676
A.4 Distributions for large losses	677
A.4.1 Extreme value distributions	677
A.4.2 Generalized Pareto distributions	678
A.5 Other distributions	678
A.6 Distributions with finite support	680
Appendix B: An inventory of discrete distributions	683
B.1 Introduction	683
B.2 The $(a, b, 0)$ class	684
B.3 The $(a, b, 1)$ class	685
B.3.1 The zero-truncated subclass	685
B.3.2 The zero-modified subclass	687
B.4 The compound class	688
B.4.1 Some compound distributions	688
B.5 A hierarchy of discrete distributions	690
Appendix C: Frequency and severity relationships	691
Appendix D: The recursive formula	693
Appendix E: Discretization of the severity distribution	695
E.1 The method of rounding	695
E.2 Mean preserving	696
E.3 Undiscretization of a discretized distribution	696
Appendix F: Numerical optimization and solution of systems of equations	699
F.1 Maximization using Solver	699
F.2 The simplex method	704
F.3 Using Excel® to solve equations	705
References	709
Index	719