

"This is an extraordinary book, really one of a kind. Written by two supreme experts, but aimed at the level of an undergraduate or a curious amateur, it emphasizes the really powerful ideas, with the bare minimum of math notation and the maximum number of elegant and suggestive visuals. The authors explain why this legendary problem is so beautiful, why it is difficult, and why you should care."

- Will Hearst, Hearst Corporation

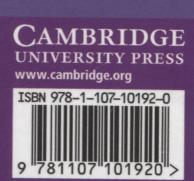
"This book is a soaring ride, starting from the simplest ideas and ending with one of the deepest unsolved problems of mathematics. Unlike in many popular math books puffed up with anecdotal material, the authors here treat the reader as seriously interested in prime numbers and build up the real math in four stages with compelling graphical demonstrations revealing in deeper and deeper ways the hidden music of the primes. If you have ever wondered why so many mathematicians are obsessed with primes, here's the real deal."

- David Mumford, Brown University

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