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"This book is a soaring ride, starting from the simplest ideas and ending with one of the deepest unsolved problems of mathematics. Unlike in many popular math books puffed up with anecdotal material, the authors here treat the reader as seriously interested in prime numbers and build up the real math in four stages with compelling graphical demonstrations revealing in deeper and deeper ways the hidden music of the primes. If you have ever wondered why so many mathematicians are obsessed with primes, here's the real deal."

– David Mumford, Brown University

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PART I. The Riemann Hypothesis	1
1 Thoughts About Numbers	3
2 What are Prime Numbers?	6
3 “Named” Prime Numbers	11
4 Sieves	13
5 Questions About Primes	16
6 Further Questions About Primes	19
7 How Many Primes are There?	23
8 Prime Numbers Viewed from a Distance	28
9 Pure and Applied Mathematics	30
10 A Probabilistic First Guess	32
11 What is a “Good Approximation”?	36
12 Square Root Error and Random Walks	38
13 What is Riemann’s Hypothesis?	40
14 The Mystery Moves to the Error Term	42
15 Cesàro Smoothing	43
16 A View of $ \text{Li}(X) - \pi(X) $	45
17 The Prime Number Theorem	47
18 The Staircase of Primes	51

19	Tinkering with the Staircase of Primes	53
20	Computer Music Files and Prime Numbers	56
21	The Word “Spectrum”	62
22	Spectra and Trigonometric Sums	64
23	The Spectrum and the Staircase of Primes	66
24	To Our Readers of Part I	67
PART II. Distributions		69
25	Slopes of Graphs That Have No Slopes	71
26	Distributions	77
27	Fourier Transforms: Second Visit	82
28	Fourier Transform of Delta	85
29	Trigonometric Series	87
30	A Sneak Preview of Part III	89
PART III. The Riemann Spectrum of the Prime Numbers		95
31	On Losing No Information	97
32	From Primes to the Riemann Spectrum	99
33	How Many θ_i 's are There?	104
34	Further Questions About the Riemann Spectrum	106
35	From the Riemann Spectrum to Primes	108
PART IV. Back to Riemann		111
36	Building $\pi(X)$ from the Spectrum	113
37	As Riemann Envisioned It	119
38	Companions to the Zeta Function	125
<i>Endnotes</i>		129
<i>Index</i>		141