

Topics in Mathematical Modeling

K. K. TUNG

Topics in Mathematical Modeling is an introductory textbook on mathematical modeling. The book teaches how simple mathematics can help formulate and solve real problems of current research interest in a wide range of fields, including biology, ecology, computer science, geophysics, engineering, and the social sciences. Yet the prerequisites are minimal: calculus and elementary differential equations. Among the many topics addressed are HIV; plant phyllotaxis; global warming; the World Wide Web; plant and animal vascular networks; social networks; chaos and fractals; marriage and divorce; and El Niño. Traditional modeling topics such as predator–prey interaction, harvesting, and wars of attrition are also included. Most chapters begin with the history of a problem, follow with a demonstration of how it can be modeled using various mathematical tools, and close with a discussion of its remaining unsolved aspects.

Designed for a one-semester course, the book progresses from problems that can be solved with relatively simple mathematics to ones that require more sophisticated methods. The math techniques are taught as needed to solve the problem being addressed, and each chapter is designed to be largely independent to give teachers flexibility.

The book, which can be used as an over-

view and introduction to applied mathematics, is particularly suitable for sophomore, junior, and senior students in math, science, and engineering.

K. K. TUNG is Professor and Chairman of the Department of Applied Mathematics at the University of Washington. He is the author or coauthor of more than eighty research papers in atmospheric sciences and applied mathematics, and editor or chief editor of two journals in these fields.

“This book has a refreshing style that should appeal to undergraduates. Indeed, the author has produced a textbook that might well achieve his goal of teaching applied mathematics without those being taught noticing!”

—Andrew Wathen, *University of Oxford*

“With courses in mathematical modeling getting ever more popular, this book will make a valuable addition to the subject. It deals with topics that should be appealing even to students not majoring in math or science, and the level of mathematical sophistication is carefully increased throughout the book.”

—Henrik Kalisch, *University of Bergen, Norway*

PRINCETON UNIVERSITY PRESS
press.princeton.edu

Jacket illustration by Matthew Harding with computer modeling assistance from Mark Woolston
Cover design by Carmina Alvarez-Gaffin

ISBN-10: 0-691-11642-3
ISBN-13: 978-0-691-11642-6



9 780691 116426

Contents

Preface	xiii
---------	------

1 Fibonacci Numbers, the Golden Ratio, and Laws of Nature?

1.1 Leonardo Fibonacci	1
1.2 The Golden Ratio	7
1.3 The Golden Rectangle and Self-Similarity	10
1.4 Phyllotaxis	12
1.5 Pinecones, Sunflowers, and Other Seed Heads	15
1.6 The Hofmeister Rule	17
1.7 A Dynamical Model	20
1.8 Concluding Remarks	21
1.9 Exercises	22

2 Scaling Laws of Life, the Internet, and Social Networks

2.1 Introduction	27
2.2 Law of Quarter Powers	27
2.3 A Model of Branching Vascular Networks	30
2.4 Predictions of the Model	35
2.5 Complications and Modifications	36
2.6 The Fourth Fractal Dimension of Life	38
2.7 Zipf's Law of Human Language, of the Size of Cities, and Email	39
2.8 The World Wide Web and the Actor's Network	42
2.9 Mathematical Modeling of Citation Network and the Web	44
2.10 Exercises	47

3 Modeling Change One Step at a Time

3.1 Introduction	54
3.2 Compound Interest and Mortgage Payments	54
Your Bank Account	54
Your Mortgage Payments, Monthly Interest Compounding	56
Your Mortgage Payments, Daily Interest Compounding	57
3.3 Some Examples	58
3.4 Compounding Continuously	58
Continuous Compounding	59
Double My Money: "Rule of 72," or Is It "Rule of 69"?	60

3.5	Rate of Change	62
	Continuous Change	63
3.6	Chaotic Bank Balances	63
3.7	Exercises	65

4 **Differential Equation Models: Carbon Dating, Age of the Universe, HIV Modeling**

4.1	Introduction	68
4.2	Radiometric Dating	68
4.3	The Age of Uranium in Our Solar System	70
4.4	The Age of the Universe	71
4.5	Carbon Dating	74
4.6	HIV Modeling	77
4.7	Exercises	79

5 **Modeling in the Physical Sciences, Kepler, Newton, and Calculus**

5.1	Introduction	84
5.2	Calculus, Newton, and Leibniz	87
5.3	Vector Calculus Needed	88
5.4	Rewriting Kepler's Laws Mathematically	90
5.5	Generalizations	93
5.6	Newton and the Elliptical Orbit	95
5.7	Exercises	96

6 **Nonlinear Population Models: An Introduction to Qualitative Analysis Using Phase Planes**

6.1	Introduction	98
6.2	Population Models	98
6.3	Qualitative Analysis	100
6.4	Harvesting Models	101
6.5	Economic Considerations	103
6.6	Depensation Growth Models	104
6.7	Comments	108
6.8	Exercises	108

7 **Discrete Time Logistic Map, Periodic and Chaotic Solutions**

7.1	Introduction	113
	Logistic Growth for Nonoverlapping Generations	114
7.2	Discrete Map	115
7.3	Nonlinear Solution	117

7.4	Sensitivity to Initial Conditions	120
7.5	Order Out of Chaos	121
7.6	Chaos Is Not Random	122
7.7	Exercises	122

8 Snowball Earth and Global Warming

8.1	Introduction	126
8.2	Simple Climate Models	128
	Incoming Solar Radiation	129
	Albedo	130
	Outward Radiation	130
	Ice Dynamics	132
	Transport	132
	The Model Equation	133
8.3	The Equilibrium Solutions	134
	Ice-Free Globe	135
	Ice-Covered Globe	136
	Partially Ice-Covered Globe	137
	Multiple Equilibria	138
8.4	Stability	139
	The Slope-Stability Theorem	140
	The Stability of the Ice-Free and Ice-Covered Globes	141
	Stability and Instability of the Partially Ice-Covered Globe	141
	How Does a Snowball Earth End?	143
8.5	Evidence of a Snowball Earth and Its Fiery End	144
8.6	The Global Warming Controversy	146
8.7	A Simple Equation for Climate Perturbation	150
8.8	Solutions	153
	Equilibrium Global Warming	153
	Time-Dependent Global Warming	154
	Thermal Inertia of the Atmosphere–Ocean System	155
8.9	Exercises	157

9 Interactions: Predator–Prey, Spraying of Pests, Carnivores in Australia

9.1	Introduction	161
9.2	The Nonlinear System and Its Linear Stability	162
9.3	Lotka–Volterra Predator–Prey Model	165
	Linear Analysis	167
	Nonlinear Analysis	170
9.4	Harvesting of Predator and Prey	172
	Indiscriminate Spraying of Insects	173

9.5	The Case of the Missing Large Mammalian Carnivores	173
9.6	Comment	176
9.7	More Examples of Interactions	178
9.8	Exercises	182

10 Marriage and Divorce

10.1	Introduction	191
10.2	Mathematical Modeling	195
	Self-interaction	196
	Marital Interactions	197
10.3	Data	198
10.4	An Example of a Validating Couple	199
10.5	Why Avoiding Conflicts Is an Effective Strategy in Marriage	201
10.6	Terminology	202
10.7	General Equilibrium Solutions	203
10.8	Conclusion	206
10.9	Assignment	206
10.10	Exercises	210

11 Chaos in Deterministic Continuous Systems, Poincaré and Lorenz

11.1	Introduction	212
11.2	Henri Poincaré	212
11.3	Edward Lorenz	214
11.4	The Lorenz Equations	216
11.5	Comments on Lorenz Equations as a Model of Convection	224
11.6	Chaotic Waterwheel	225
11.7	Exercises	226

12 El Niño and the Southern Oscillation

12.1	Introduction	229
12.2	Bjerknes' Hypothesis	231
12.3	A Simple Mathematical Model of El Niño	233
	The Atmosphere	233
	Air-Sea Interaction	234
	Ocean Temperature Advection	235
12.4	Other Models of El Niño	239
12.5	Appendix: The Advection Equation	240
12.6	Exercises	241

13	Age of the Earth: Lord Kelvin's Model	
13.1	Introduction	243
13.2	The Heat Conduction Problem	245
13.3	Numbers	250
13.4	Exercises	251
14	Collapsing Bridges: Broughton and Tacoma Narrows	
14.1	Introduction	254
14.2	Marching Soldiers on a Bridge: A Simple Model	254
	Resonance	259
	A Different Forcing Function	260
14.3	Tacoma Narrows Bridge	261
	Assignment	262
14.4	Exercises	262
	APPENDIX A: Differential Equations and Their Solutions	
A.1	First- and Second-Order Equations	267
A.2	Nonhomogeneous Ordinary Differential Equations	273
	First-Order Equations	273
	Second-Order Equations	275
A.3	Summary of ODE Solutions	277
A.4	Exercises	278
A.5	Solutions to Exercises	279
	APPENDIX B: MATLAB Codes	
B.1	MATLAB Codes for Lorenz Equations	282
B.2	MATLAB Codes for Solving Vallis's Equations	284
	Bibliography	287
	Index	293