

LONDON MATHEMATICAL SOCIETY

LECTURE NOTE SERIES

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Graded Rings and Graded Grothendieck Groups

Roozbeh Hazrat

This study of graded rings includes the first systematic account of the graded Grothendieck group; a powerful and crucial invariant in algebra which has recently been adopted to classify the Leavitt path algebras.

The book begins with a concise introduction to the theory of graded rings and then focuses in more detail on Grothendieck groups, Morita theory, Picard groups and K -theory. The author extends known results in the nongraded case to the graded setting and gathers together important results which are currently scattered throughout the literature. The book is suitable for advanced undergraduate and graduate students as well as researchers in ring theory.

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