

Michel Grabisch

Set Functions, Games and Capacities in Decision Making

Michel Grabisch has fully succeeded in transforming his deep knowledge of the field into a very rich and quite readable text.

Professor Denis Bouyssou, Université Paris Dauphine

This monograph provides a much-needed perspective on a rich collection of structural properties of set functions which have been investigated for more than 60 years, but which have never been collected in a single volume. By establishing these results in a clear and rigorous mathematical framework, the author makes them accessible to a broad audience. His book should rapidly become a classic reference in the field.

Professor Yves Crama, University of Liège and HEC Liège

This book is an excellent compendium of technical notions pertaining to set functions, that are relevant in uncertainty management, game theory, and multicriteria decision-making. It will be a very useful companion to any student or researcher willing to investigate these fields beyond standard probability theory and simple additive methods.

Professor Didier Dubois, IRIT, Université Paul Sabatier, Toulouse

In this book, written by a leading scholar, the readers can finally find a unified and comprehensive study of set functions. It is a much needed, beautiful, book that will be most useful to the cognoscenti, who will appreciate its depth and breadth, as well as to the readers who desire to learn about this fundamental topic.

Professor Massimo Marinacci, Università Bocconi

Nonadditive set functions are used in many optimization and decision theories (uncertain, fuzzy, multi-criteria, strategic, AI, combinatorial, statistical, ...), many developing the same results independently. The only person knowing all these fields brought it all together and improved it brilliantly in this well-focused book. It is a must for anyone using set functions.

Professor Peter P. Wakker, Erasmus University Rotterdam

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