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Lorenzo Mangani received his Ph.D. from the University of Florence in 2010, where he worked on the development of a state-of-the-art turbo machinery code in OpenFOAM® for heat transfer and combustion analysis. After three years of postdoc work, he joined the Lucerne University of Applied Sciences and Arts as senior research and chief engineer for CFD simulations. Since 2014, he is working as an associate professor at the Fluid-Mechanics and Hydro-machines Department, where he manages a variety of projects with industrial partners aimed at developing advanced and novel CFD tools. His research interests include pressure- and density-based solvers, segregated and fully coupled algorithms, fluid-structure interactions (FSI), turbulence, and conjugate heat transfer modeling.

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