

Contents

1	Basic Homotopy	1
1.1	Introduction	1
1.2	Spaces, Maps, Products, and Wedges	2
1.3	Homotopy I	3
1.4	Homotopy II	7
1.5	CW Complexes	18
1.6	Why Study Homotopy Theory?	29
	Exercises	31
2	H-Spaces and Co-H-Spaces	35
2.1	Introduction	35
2.2	H-Spaces and Co-H-Spaces	36
2.3	Loop Spaces and Suspensions	44
2.4	Homotopy Groups I	50
2.5	Moore Spaces and Eilenberg–Mac Lane Spaces	60
2.6	Eckmann–Hilton Duality I	67
	Exercises	70
3	Cofibrations and Fibrations	75
3.1	Introduction	75
3.2	Cofibrations	75
3.3	Fibrations	83
3.4	Examples of Fiber Bundles	93
3.5	Replacing a Map by a Cofiber or Fiber Map	104
	Exercises	109
4	Exact Sequences	115
4.1	Introduction	115
4.2	The Coexact and Exact Sequence of a Map	116
4.3	Actions and Coactions	125
4.4	Operations	130

4.5	Homotopy Groups II	135
	Exercises	150
5	Applications of Exactness	155
5.1	Introduction	155
5.2	Universal Coefficient Theorems	156
5.3	Homotopical Cohomology Groups	160
5.4	Applications to Fiber and Cofiber Sequences	163
5.5	The Operation of the Fundamental Group	169
5.6	Calculation of Homotopy Groups	177
	Exercises	190
6	Homotopy Pushouts and Pullbacks	195
6.1	Introduction	195
6.2	Homotopy Pushouts and Pullbacks I	196
6.3	Homotopy Pushouts and Pullbacks II	207
6.4	Theorems of Serre, Hurewicz, and Blakers–Massey	214
6.5	Eckmann–Hilton Duality II	225
	Exercises	227
7	Homotopy and Homology Decompositions	233
7.1	Introduction	233
7.2	Homotopy Decompositions of Spaces	234
7.3	Homology Decompositions of Spaces	247
7.4	Homotopy and Homology Decompositions of Maps	254
	Exercises	264
8	Homotopy Sets	267
8.1	Introduction	267
8.2	The Set $[X, Y]$	267
8.3	Category	270
8.4	Loop and Group Structure in $[X, Y]$	275
	Exercises	279
9	Obstruction Theory	283
9.1	Introduction	283
9.2	Obstructions Using Homotopy Decompositions	284
9.3	Lifts and Extensions	288
9.4	Obstruction Miscellany	291
	Exercises	296
A	Point-Set Topology	299
B	The Fundamental Group	303
C	Homology and Cohomology	307

D Homotopy Groups of the n -Sphere 312

E Homotopy Pushouts and Pullbacks 314

F Categories and Functors 323

Hints to Some of the Exercises 329

References 335

Index 339

1.1 Introduction

In this book we study topological spaces and continuous functions from one topological space to another. In algebraic topology these objects are studied by developing algebraic invariants to them. We assign groups, rings, vector spaces, or other algebraic objects to topological spaces and we assign homomorphisms of these objects to continuous functions. A useful equivalence relation called homotopy on the set of continuous functions from one topological space into another naturally arises in the study of these invariants. By investigating this relation we obtain interesting, deep, and sometimes surprising information about topological spaces and continuous functions and their algebraic counterparts. In addition, the results of homotopy lead to new algebraic invariants for topological spaces and continuous functions.

We begin this chapter by recalling the notion of homotopy and its properties. We then introduce a new way of describing homotopy called the homotopy equivalence relation and continuity. We describe the homotopy equivalence relation and continuity in terms of continuous functions. In section 1.5 we introduce CW complexes and derive many of their properties. These spaces are of fundamental importance in homotopy theory. They are broad enough to include most of the spaces that are of interest to topologists. But they are sufficiently restricted so that many important results which do not hold for arbitrary spaces such as the homotopy extension property and Whitehead's theorem hold for CW complexes. Furthermore, the definition of a CW complex is recursive, and so recursive arguments can often be used. From Chapter 2 on we assume that all spaces under consideration have the homotopy type of CW complexes. In Section 1.6 we briefly discuss some reasons for studying homotopy theory by indicating some applications of homotopy theory to other areas of topology.