

'Picturing Quantum Processes is a lively and refreshing romp through the author's diagrammatic and categorical approach to quantum processes. I recommend this book with no lower age limit required!'

Louis Kauffman, *University of Illinois*

'This book develops from scratch the category theoretic, and diagrammatic, language for quantum theory, especially quantum processes. It is a remarkable achievement: vigorous, crystal-clear, complete –and a delight to read.'

Jeremy Butterfield, *University of Cambridge*

The unique features of the quantum world are explained in this book through the language of diagrams, setting out an innovative visual method for presenting complex theories. Requiring only basic mathematical literacy this book employs a unique formalism that builds an intuitive understanding of quantum features while eliminating the need for complex calculations. This entirely diagrammatic presentation of quantum theory represents the culmination of 10 years of research, uniting classical techniques in linear algebra and Hilbert spaces with cutting-edge developments in quantum computation and foundations.

Written in an entertaining and user-friendly style and including more than 100 exercises, this book is an ideal first course in quantum theory, foundations, and computation for students from undergraduate to PhD level, as well as an opportunity for researchers from a broad range of fields, from physics to biology, linguistics, and cognitive science, to discover a new set of tools for studying processes and interaction.

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