CONTENTS

1. The System of Numbers: An Overview 1

1.1 From natural to real numbers 3

1.2 Imaginary numbers 9

1.3 Polynomials and transcendental numbers 11

1.4 Cardinals and ordinals 15

2. Writing Numbers—Now and Back Then 17

2.1 Writing numbers nowadays: positional and decimal 17

2.2 Writing numbers back then: Egypt, Babylon and Greece 24

3. Numbers and Magnitudes in the Greek Mathematical Tradition 31

3.1 Pythagorean numbers 32

3.2 Ratios and proportions 35

3.3 Incommensurability 39

3.4 Eudoxus' theory of proportions 42

3.5 Greek fractional numbers 45

3.6 Comparisons, not measurements 47

3.7 A unit length 50

Appendix 3.1 The incommensurability of $\sqrt{2}$. Ancient and modern proofs 52

Appendix 3.2 Eudoxus' theory of proportions in action 55

Appendix 3.3 Euclid and the area of the circle 59

4. Construction Problems and Numerical Problems in the Greek Mathematical Tradition 63

4.1 The arithmetic books of the *Elements* 64

4.2 Geometric algebra? 66

4.3 Straightedge and compass 67

4.4 Diophantus' numerical problems 71

4.5 Diophantus' reciprocals and fractions 78

4.6 More than three dimensions 80

Appendix 4.1 Diophantus' solution of Problem V.9 in *Arithmetica* 83

5. Numbers in the Tradition of Medieval Islam 87
5.1 Islamicate science in historical perspective 88
5.2 Al-Khwārizmī and numerical problems with squares 90
5.3 Geometry and certainty 94
5.4 *Al-jabr wa'l-muqābala* 97
5.5 Al-Khwārizmī, numbers and fractions 100
5.6 Abū Kāmil's numbers at the crossroads of two traditions 103
5.7 Numbers, fractions and symbolic methods 107
5.8 Al-Khayyām and numerical problems with cubes 111
5.9 Gersonides and problems with numbers 116
Appendix 5.1 The quadratic equation. Derivation of the algebraic formula 120
Appendix 5.2 The cubic equation. Khayyam's geometric solution 121

6. Numbers in Europe from the Twelfth to the Sixteenth Centuries 125
6.1 Fibonacci and Hindu–Arabic numbers in Europe 128
6.2 Abbacus and coss traditions in Europe 129
6.3 Cardano's *Great Art of Algebra* 138
6.4 Bombelli and the roots of negative numbers 146
6.5 Euclid's *Elements* in the Renaissance 149
Appendix 6.1 Casting out nines 150

7. Number and Equations at the Beginning of the Scientific Revolution 155
7.1 Viète and the new art of analysis 157
7.2 Stevin and decimal fractions 163
7.3 Logarithms and the decimal system of numeration 167
Appendix 7.1 Napier's construction of logarithmic tables 171

8. Number and Equations in the Works of Descartes, Newton and their Contemporaries 175
8.1 Descartes' new approach to numbers and equations 176
8.2 Wallis and the primacy of algebra 182
8.3 Barrow and the opposition to the primacy of algebra 187
8.4 Newton's *Universal Arithmetick* 190
Appendix 8.1 The quadratic equation. Descartes' geometric solution 196
Appendix 8.2 Between geometry and algebra in the seventeenth century: The case of Euclid's *Elements* 198

9. New Definitions of Complex Numbers in the Early Nineteenth Century 207
9.1 Numbers and ratios: giving up metaphysics 208
9.2 Euler, Gauss and the ubiquity of complex numbers 209
9.3 Geometric interpretations of the complex numbers 212
9.4 Hamilton's formal definition of complex numbers 215
9.5 Beyond complex numbers 217
9.6 Hamilton's discovery of quaternions 220

10. "What Are Numbers and What Should They Be?" Understanding Numbers in the Late Nineteenth Century 223
10.1 What are numbers? 224
10.2 Kummer's ideal numbers 225
10.3 Fields of algebraic numbers 228
10.4 What should numbers be? 231
10.5 Numbers and the foundations of calculus 234
10.6 Continuity and irrational numbers 237
Appendix 10.1 Dedekind's theory of cuts and Eudoxus' theory of proportions 243
Appendix 10.2 IVT and the fundamental theorem of calculus 245

11. Exact Definitions for the Natural Numbers: Dedekind, Peano and Frege 249
11.1 The principle of mathematical induction 250
11.2 Peano's postulates 251
11.3 Dedekind's chains of natural numbers 257
11.4 Frege's definition of cardinal numbers 259
Appendix 11.1 The principle of induction and Peano's postulates 262

12. Numbers, Sets and Infinity. A Conceptual Breakthrough at the Turn of the Twentieth Century 265
12.1 Dedekind, Cantor and the infinite 266
12.2 Infinities of various sizes 269
12.3 Cantor's transfinite ordinals 277
12.4 Troubles in paradise 280
Appendix 12.1 Proof that the set of algebraic numbers is countable 287

13. Epilogue: Numbers in Historical Perspective 291

References and Suggestions for Further Reading 295
Name Index 303
Subject Index 306