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The Theory of Nilpotent Groups

This monograph presents both classical and recent results in the theory of nilpotent groups and provides a self-contained, comprehensive reference on the topic. While the theorems and proofs included can be found throughout the existing literature, this is the first book to collect them in a single volume. Details omitted from the original sources, along with additional computations and explanations, have been added to foster a stronger understanding of the theory of nilpotent groups and the techniques commonly used to study them. Topics discussed include collection processes, normal forms and embeddings, isolators, extraction of roots, P-localization, dimension subgroups and Lie algebras, decision problems, and nilpotent groups of automorphisms. Requiring only a strong undergraduate or beginning graduate background in algebra, graduate students and researchers in mathematics will find *The Theory of Nilpotent Groups* to be a valuable resource.

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