

Contents

| | |
|--|-----|
| Preface | xi |
| Introduction | 1 |
| Part 1. First Order Conditions | 5 |
| Chapter 1. Theory of a Weak Minimum for the Problem on a Fixed Time Interval | 7 |
| 1. Problems of the calculus of variations | 7 |
| 2. The problem on a fixed time interval. Necessary conditions for a weak extremum | 8 |
| 3. Two examples | 14 |
| 4. Weak extremals | 18 |
| Chapter 2. Theory of the Maximum Principle | 23 |
| 5. Formulation of the maximum principle for the problem of § 1 | 23 |
| 6. Invariance of Pontryagin's convergence under the change of the independent variable | 26 |
| 7. Proof of the maximum principle | 32 |
| 8. Expansion formulas | 40 |
| Chapter 3. Extremals and the Hamiltonian of a Control System | 43 |
| 9. Extremals | 43 |
| 10. Solutions of a Hamiltonian system and extremals | 47 |
| 11. Examples | 51 |
| 11.1. Time-optimal control problems | 51 |
| 11.2. Two optimal control problems on a fixed time interval | 55 |
| 11.3. Isoperimetric problem | 63 |
| 11.4. Singular extremals | 66 |
| 12. Convexification of the right-hand side of a control system (sliding modes) | 69 |
| Chapter 4. Hamilton–Jacobi Equation and Field Theory | 87 |
| 13. The Hamilton–Jacobi equation and sufficient conditions for a strong minimum | 87 |
| 14. Solutions of the Hamilton–Jacobi equation and extremals | 92 |
| 15. The field of extremals | 105 |
| 15.1. The general field theory | 105 |
| 15.2. A linear control system | 112 |

| | |
|---|-----|
| 15.3. The field in the problem with a general control system and a primitive endpoint part | 116 |
| 15.4. Isoperimetric problem | 121 |
| Chapter 5. Transformations of Problems and Invariance of Extremals | 125 |
| 16. Invariance of extremals | 125 |
| 16.1. Change of variables | 125 |
| 16.2. Change of the independent variable | 134 |
| 16.3. Passage to a parametric form | 140 |
| 17. Calculus of variations problems with pointwise equality constraints | 140 |
| 18. Problems with pointwise mixed state-control equality and inequality constraints | 148 |
| Part 2. Quadratic Conditions | 153 |
| Chapter 1. Quadratic Conditions and Conjugate Points for Broken Extremals | 155 |
| 1. Quadratic conditions in the simplest problem of the calculus of variations | 155 |
| 1.1. The setup and assumptions | 155 |
| 1.2. Minimum on a set of sequences | 157 |
| 1.3. First order conditions | 158 |
| 1.4. An additional condition of the Weierstrass-Erdmann type | 160 |
| 1.5. The Legendre condition | 162 |
| 1.6. Quadratic conditions for a weak minimum | 162 |
| 1.7. Quadratic conditions for a Pontryagin minimum | 163 |
| 1.8. Sufficient conditions for a bounded-strong minimum | 166 |
| 1.9. θ -weak minimum | 167 |
| 2. Conjugate points and conditions for positive definiteness of the quadratic form | 168 |
| 2.1. Passage of the quadratic form through zero: the classical Jacobi condition | 168 |
| 2.2. Positive definiteness of Ω on G_2 | 177 |
| 3. Conditions for nonnegativeness of the quadratic form | 186 |
| 3.1. Nonnegativeness of ω on E_2 | 186 |
| 3.2. Nonnegativeness of Ω on G_2 | 187 |
| 3.3. Abstract model | 189 |
| 3.4. Nonnegativeness of Ω on G_2 (continued) | 190 |
| 4. Investigation of a broken extremal by means of conjugate points theory: an example | 196 |
| Chapter 2. Quadratic Conditions for a Pontryagin Minimum and Sufficient Conditions for a Strong Minimum: Proofs | 211 |
| 5. Higher orders, γ -conditions, and constant C_γ | 211 |
| 5.1. Order γ | 211 |
| 5.2. γ -conditions | 214 |
| 5.3. Constant C_γ | 215 |
| 6. Expansion of the integral functional on local sequences of variations | 217 |
| 6.1. The structure of local sequences. The main lemma | 217 |
| 6.2. Representation for the increment δF on local sequences | 222 |

| | |
|---|-----|
| 6.3. Proof of Lemma 6.1, continued | 223 |
| 6.4. Proof of Lemma 6.1, concluded | 225 |
| 6.5. Proof of Theorem 1.1 | 228 |
| 6.6. Estimation of $\ \delta x\ _C$ on local sequences | 228 |
| 7. Upper bound for C_γ | 229 |
| 7.1. The constant C_γ^{loc} | 229 |
| 7.2. Extension of the set $\Pi^{\text{loc}}(E)$ | 230 |
| 7.3. Canonical representation for sequences in $\Pi_{O(\gamma)}^{\text{loc}}$ | 233 |
| 7.4. Passage to sequences with $\delta v = 0$ | 234 |
| 7.5. Passage to sequences with discontinuous state components | 235 |
| 7.6. The set of sequences S^3 | 241 |
| 7.7. The sets of sequences S^4 and S^5 | 241 |
| 7.8. The space $Z(\theta)$ and subspace G | 243 |
| 7.9. Passage to G_2 | 244 |
| 7.10. The CT-strict maximum principle | 245 |
| 8. Lower bound for C_γ | 249 |
| 8.1. Extension of the set $\Pi(E)$ | 249 |
| 8.2. Passage to local sequences | 249 |
| 8.3. Simplifications in the definition of $C_\gamma(\Phi, \Pi_{o(\sqrt{\gamma})}^{\text{loc}})$ | 256 |
| 8.4. Application of the Legendre condition | 258 |
| 8.5. Passage to sequences with discontinuous state components | 260 |
| 8.6. Condition $D^k(H) \geq 2C_\Gamma(H)$ | 261 |
| 8.7. Passage to the equality in the differential constraint | 265 |
| 8.8. The final lower bound for C_γ . The result of deciphering | 266 |
| 8.9. Proof of Theorem 5.1 | 267 |
| 8.10. Proof of Theorem 5.2 | 267 |
| 9. Sufficient conditions for bounded-strong and strong minima in the simplest problem of the calculus of variations | 271 |
| 9.1. Sufficient conditions for a bounded-strong minimum. Proof of Theorem 1.5 | 271 |
| 9.2. γ -sufficiency on $\bar{\Pi}^S$ | 275 |
| 9.3. Sufficient conditions for a strong minimum | 275 |
| Chapter 3. Quadratic Conditions in the General Problem of the Calculus of Variations and Related Optimal Control Problems | 283 |
| 10. Formulation of quadratic conditions in the general problem of the calculus of variations | 283 |
| 10.1. The problem setting and assumptions | 283 |
| 10.2. First order conditions; the set M_0 | 284 |
| 10.3. Critical cone | 286 |
| 10.4. The quadratic form | 287 |
| 10.5. Quadratic necessary condition | 289 |
| 10.6. Strong and bounded-strong minima | 290 |
| 10.7. The strict maximum principle and strictly Legendrian elements | 291 |
| 11. Quadratic conditions in the general problem of the calculus of variations on a fixed time interval | 293 |
| 11.1. Formulation of quadratic conditions | 293 |

| | |
|---|-----|
| 11.2. γ -sufficiency | 295 |
| 11.3. Discussion of the proofs of quadratic conditions | 296 |
| 12. Quadratic conditions in problems that are linear in control | 298 |
| 12.1. Linear in control problem on a fixed time interval | 298 |
| 12.2. Quadratic conditions in the linear in control problem on a non-fixed time interval | 302 |
| 12.3. Quadratic conditions for a piecewise constant control | 303 |
| 12.4. Quadratic conditions in the minimum time problem for a system linear in control | 308 |
| 13. Quadratic conditions in time-optimal control problems for linear systems with constant coefficients | 310 |
| 13.1. Problem setting, maximum principle, and simple sufficient conditions | 310 |
| 13.2. Quadratic necessary condition | 314 |
| 13.3. Quadratic sufficient condition | 318 |
| 13.4. Nonemptiness of the set Ξ | 322 |
| 13.5. The case where M_0 is a singleton | 323 |
| Chapter 4. Investigation of Extremals by Quadratic Conditions: Examples | 325 |
| 14. Time-optimal control problems for linear systems with constant coefficients | 325 |
| 14.1. Two-dimensional chain | 325 |
| 14.2. Oscillating system | 328 |
| 14.3. Three-dimensional chain | 331 |
| 14.4. Oscillating system with an additional integral constraint on the control | 336 |
| 15. Investigation of extremals in nonlinear systems | 341 |
| 15.1. Saw-shaped extremals | 341 |
| 15.2. Isoperimetric problem | 344 |
| 15.3. M_0 consisting of many elements | 349 |
| 16. Appendix | 361 |
| 16.1. Integrals of convex combinations | 361 |
| 16.2. A property of systems that are linear in control | 366 |
| 16.3. Lyusternik's theorem | 368 |
| 16.4. Condition for inconsistency of a system of linear inequalities | 368 |
| Bibliography | 371 |