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Preface

BYRON J. PETERSON

This is a book about the foundations of mathematics—a topic once of great interest to outstanding mathematicians, such as Dedekind, Peano, and Hilbert, but today sadly neglected. This neglect is unfortunate for several reasons.

- As mathematics splits into more and more specialties, the need for a unifying viewpoint becomes more acute.
- Research is useful not only mathematics but also the neighboring disciplines of computer science and physics.
- Recent advances in mathematical logic throw new light on the foundations of analysis, and on the elusive concept of mathematical "depth."

This book gives at the last point in particular by focusing on the topic of *reverse mathematics*.

As its name suggests, reverse mathematics looks at the concept of *proof* in the opposite to normal direction. Instead of seeking the consequences of given axioms, it asks the axioms needed to prove given theorems. This is actually an old idea, at least in the foundations of geometry. From the time of Euclid until the nineteenth century it was a burning question whether the parallel axiom was needed to prove theorems such as the Pythagorean theorem. We review the history of the parallel axiom in chapter 1 of this book, as a case study in reverse mathematical ideas, together with the similar story of the axiom of choice in set theory.

Although both these axioms illustrate the idea of reverse mathematics, the subject as it is understood today lies mostly in a narrow but important region between geometry and set theory: the theory of real numbers, which is the foundation of calculus, analysis, and most of mathematical physics. (Reverse mathematics has also made interesting contributions to algebra, combinatorics, and topology which we mention more briefly.)