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Objectives of Analyzing Multiple Time Series

In making choices between alternative courses of action, decision makers at

observations are available for a variable of interest and the data from the past contain information about the future development of a variable, it is desirable to use as forecast some function of the data collected in the past. For instance, in forecasting the monthly unemployment rate, from past experience a forecaster may know that in some country or region a high unemployment rate in one month tends to be followed by a high rate in the next month. In other words, the rate changes only gradually. Assuming that the tendency persists in future periods, forecasts can be based on current and past data.

Formally, this approach to forecasting may be expressed as follows. Let y_t denote the value of the variable of interest in period t . Then a forecast for period $T + h$, made at the end of period T , may have the form

$$\hat{y}_{T+h} = f(y_T, y_{T-1}, \dots), \quad (1.1.1)$$

where $f(\cdot)$ denotes some suitable function of the past observations $y_T, y_{T-1}, \dots$. For the moment it is left open how many past observations enter into the forecast. One major goal of univariate time series analysis is to specify suitable forms of functions $f(\cdot)$. In many applications, linear functions have been used so that, for example,

$$\hat{y}_{T+h} = \alpha_0 + \alpha_1 y_T + \alpha_2 y_{T-1} + \dots + \alpha_h y_{T-h+1}.$$

In dealing with economic variables, often the value of one variable is not only related to its predecessors in time but, in addition, it depends on past values of other variables. For instance, household consumption expenditures may depend on variables such as income, interest rates, and investment expenditures. If all these variables are related to the consumption expenditures