

Contents

1	Smooth Manifolds	1
	Topological Manifolds	2
	Smooth Structures	10
	Examples of Smooth Manifolds	17
	Manifolds with Boundary	24
	Problems	29
2	Smooth Maps	32
	Smooth Functions and Smooth Maps	32
	Partitions of Unity	40
	Problems	48
3	Tangent Vectors	50
	Tangent Vectors	51
	The Differential of a Smooth Map	55
	Computations in Coordinates	60
	The Tangent Bundle	65
	Velocity Vectors of Curves	68
	Alternative Definitions of the Tangent Space	71
	Categories and Functors	73
	Problems	75
4	Submersions, Immersions, and Embeddings	77
	Maps of Constant Rank	77
	Embeddings	85
	Submersions	88
	Smooth Covering Maps	91
	Problems	95
5	Submanifolds	98
	Embedded Submanifolds	98
	Immersed Submanifolds	108

Restricting Maps to Submanifolds	112
The Tangent Space to a Submanifold	115
Submanifolds with Boundary	120
Problems	123
6 Sard's Theorem	125
Sets of Measure Zero	125
Sard's Theorem	129
The Whitney Embedding Theorem	131
The Whitney Approximation Theorems	136
Transversality	143
Problems	147
7 Lie Groups	150
Basic Definitions	151
Lie Group Homomorphisms	153
Lie Subgroups	156
Group Actions and Equivariant Maps	161
Problems	171
8 Vector Fields	174
Vector Fields on Manifolds	174
Vector Fields and Smooth Maps	181
Lie Brackets	185
The Lie Algebra of a Lie Group	189
Problems	199
9 Integral Curves and Flows	205
Integral Curves	206
Flows	209
Flowouts	217
Flows and Flowouts on Manifolds with Boundary	222
Lie Derivatives	227
Commuting Vector Fields	231
Time-Dependent Vector Fields	236
First-Order Partial Differential Equations	239
Problems	245
10 Vector Bundles	249
Vector Bundles	249
Local and Global Sections of Vector Bundles	255
Bundle Homomorphisms	261
Subbundles	264
Fiber Bundles	268
Problems	268

11	The Cotangent Bundle	272
	Covectors	272
	The Differential of a Function	280
	Pullbacks of Covector Fields	284
	Line Integrals	287
	Conservative Covector Fields	292
	Problems	299
12	Tensors	304
	Multilinear Algebra	305
	Symmetric and Alternating Tensors	313
	Tensors and Tensor Fields on Manifolds	316
	Problems	324
13	Riemannian Metrics	327
	Riemannian Manifolds	327
	The Riemannian Distance Function	337
	The Tangent–Cotangent Isomorphism	341
	Pseudo-Riemannian Metrics	343
	Problems	344
14	Differential Forms	349
	The Algebra of Alternating Tensors	350
	Differential Forms on Manifolds	359
	Exterior Derivatives	362
	Problems	373
15	Orientations	377
	Orientations of Vector Spaces	378
	Orientations of Manifolds	380
	The Riemannian Volume Form	388
	Orientations and Covering Maps	392
	Problems	397
16	Integration on Manifolds	400
	The Geometry of Volume Measurement	401
	Integration of Differential Forms	402
	Stokes's Theorem	411
	Manifolds with Corners	415
	Integration on Riemannian Manifolds	421
	Densities	427
	Problems	434
17	De Rham Cohomology	440
	The de Rham Cohomology Groups	441
	Homotopy Invariance	443
	The Mayer–Vietoris Theorem	448
	Degree Theory	457

Proof of the Mayer–Vietoris Theorem	460
Problems	464
18 The de Rham Theorem	467
Singular Homology	467
Singular Cohomology	472
Smooth Singular Homology	473
The de Rham Theorem	480
Problems	487
19 Distributions and Foliations	490
Distributions and Involutivity	491
The Frobenius Theorem	496
Foliations	501
Lie Subalgebras and Lie Subgroups	505
Overdetermined Systems of Partial Differential Equations	507
Problems	512
20 The Exponential Map	515
One-Parameter Subgroups and the Exponential Map	516
The Closed Subgroup Theorem	522
Infinitesimal Generators of Group Actions	525
The Lie Correspondence	530
Normal Subgroups	533
Problems	536
21 Quotient Manifolds	540
Quotients of Manifolds by Group Actions	541
Covering Manifolds	548
Homogeneous Spaces	550
Applications to Lie Theory	555
Problems	560
22 Symplectic Manifolds	564
Symplectic Tensors	565
Symplectic Structures on Manifolds	567
The Darboux Theorem	571
Hamiltonian Vector Fields	574
Contact Structures	581
Nonlinear First-Order PDEs	585
Problems	590
Appendix A Review of Topology	596
Topological Spaces	596
Subspaces, Products, Disjoint Unions, and Quotients	601
Connectedness and Compactness	607
Homotopy and the Fundamental Group	612
Covering Maps	615

Appendix B Review of Linear Algebra	617
Vector Spaces	617
Linear Maps	622
The Determinant	628
Inner Products and Norms	635
Direct Products and Direct Sums	638
Appendix C Review of Calculus	642
Total and Partial Derivatives	642
Multiple Integrals	649
Sequences and Series of Functions	656
The Inverse and Implicit Function Theorems	657
Appendix D Review of Differential Equations	663
Existence, Uniqueness, and Smoothness	663
Simple Solution Techniques	672
References	675
Notation Index	678
Subject Index	683