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Semigroups in Complete Lattices

Quantales, Modules and Related Topics

This monograph provides a modern introduction to the theory of quantales.

First coined by C. J. Mulvey in 1986, quantales have since developed into a significant topic at the crossroads of algebra and logic, of notable interest to theoretical computer science. This book recasts the subject within the powerful framework of categorical algebra, showcasing its versatility through applications to C^* - and MV -algebras, fuzzy sets and automata. With exercises and historical remarks at the end of each chapter, this self-contained book provides readers with a valuable source of references and hints for future research.

This book will appeal to researchers across mathematics and computer science with an interest in category theory, lattice theory, and many-valued logic.

Mathematics

ISBN 978-3-319-78947-7



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