Mathematics and Its Applications

New Trends in Quantum Structures

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and

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This monograph deals with the latest results concerning different types of quantum structures. This is an interdisciplinary realm joining mathematics, logic and fuzzy reasoning with mathematical foundations of quantum mechanics, and it indicates many ways of their applications.
The book consists of seven chapters. The first four chapters are devoted to difference posets and effect algebras; MV-algebras and quantum MV-algebras, and their quotients; and to tensor product of difference posets. Chapters 5 and 6 discuss BCK-algebras with their applications. Chapter 7 addresses Loomis-Sikorski-type theorems for MV-algebras and BCK-algebras. Throughout the book, important facts and concepts are illustrated by exercises.

Audience
This book will be of interest to mathematicians, physicists, logicians, philosophers, quantum computer experts, and students interested in mathematical foundations of quantum mechanics as well as in non-commutative measure theory, o[illegible] MV-algebras, effect algebras, Hilbert space quantum m[illegible]deas.

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