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Unbounded Self-adjoint Operators on Hilbert Space

The book is a graduate text on unbounded self-adjoint operators on Hilbert space and their spectral theory with the emphasis on applications in mathematical physics (especially, Schrödinger operators) and analysis (Dirichlet and Neumann Laplacians, Sturm-Liouville operators, Hamburger moment problem). Among others, a number of advanced special topics are treated on a text book level accompanied by numerous illustrating examples and exercises. The main themes of the book are the following:

- Spectral integrals and spectral decompositions of self-adjoint and normal operators
- Perturbations of self-adjointness and of spectra of self-adjoint operators
- Forms and operators
- Self-adjoint extension theory: boundary triplets, Krein-Birman-Vishik theory of positive self-adjoint extensions.

Mathematics

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Part I Basics of Closed Operators

1	Closed and Adjoint Operators	3
1.1	Closed and Closable Operators and Their Graphs	3
1.1.1	General Notions on Linear Operators	3
1.1.2	Closed and Closable Operators	5
1.2	Adjoint Operators	8
1.3	Examples: Differential Operators	13
1.3.1	Differentiation Operators on Intervals I	13
1.3.2	Linear Partial Differential Operators	18
1.4	Invariant Subspaces and Reducing Subspaces	20
1.5	Exercises	22
2	The Spectrum of a Closed Operator	25
2.1	Regular Points and Defect Numbers of Operators	25
2.2	Spectrum and Resolvent of a Closed Operator	28
2.3	Examples: Differentiation Operators II	33
2.4	Exercises	35
3	Some Classes of Unbounded Operators	37
3.1	Symmetric Operators	37
3.2	Self-adjoint Operators	42
3.3	Accretive and Sectorial Operators	48
3.4	Normal Operators	51
3.5	Exercises	54
3.6	Notes to Part I	57

Part II Spectral Theory

4	Spectral Measures and Spectral Integrals	61
4.1	Resolutions of the Identity and Operator Stieltjes Integrals	61
4.2	Spectral Measures	66

4.2.1	Definitions and Basic Properties	66
4.2.2	Three Technical Results on Spectral Measures	70
4.3	Spectral Integrals	73
4.3.1	Spectral Integrals of Bounded Measurable Functions	74
4.3.2	Integrals of Unbounded Measurable Functions	75
4.3.3	Properties of Spectral Integrals	79
4.4	Exercises	84
5	Spectral Decompositions of Self-adjoint and Normal Operators	85
5.1	Spectral Theorem for Bounded Self-adjoint Operators	85
5.2	Spectral Theorem for Unbounded Self-adjoint Operators	89
5.3	Functional Calculus and Various Applications	91
5.4	Self-adjoint Operators with Simple Spectra	99
5.5	Spectral Theorem for Finitely Many Strongly Commuting Normal Operators	101
5.5.1	The Spectral Theorem for n -Tuples	101
5.5.2	Joint Spectrum for n -Tuples	104
5.6	Strong Commutativity of Unbounded Normal Operators	105
5.7	Exercises	111
5.8	Notes to Part II	113

Part III Special Topics

6	One-Parameter Groups and Semigroups of Operators	117
6.1	Groups of Unitaries	117
6.2	Differential Equations on Hilbert Spaces	123
6.3	Semigroups of Contractions on Banach Spaces	126
6.4	Semigroups of Contractions on Hilbert Spaces	131
6.5	Exercises	133
7	Miscellanea	137
7.1	The Polar Decomposition of a Closed Operator	137
7.2	Application to the Operator Relation $AA^* = A^*A + I$	139
7.3	The Bounded Transform of a Closed Operator	142
7.4	Analytic Vectors, Quasi-analytic Vectors, Stieltjes Vectors, and Self-adjointness of Symmetric Operators	144
7.5	Tensor Products of Operators	154
7.5.1	Tensor Product of Hilbert Spaces	154
7.5.2	Tensor Product of Operators	156
7.6	Exercises	162
7.7	Notes to Part III	164

Part IV Perturbations of Self-adjointness and Spectra

8	Perturbations of Self-adjoint Operators	167
8.1	Differential Operators with Constant Coefficients on $L^2(\mathbb{R}^d)$	167
8.2	Relatively Bounded Perturbations of Self-adjoint Operators	170

8.3	Applications to Schrödinger Operators: Self-adjointness	173
8.4	Essential Spectrum of a Self-adjoint Operator	178
8.5	Relatively Compact Perturbations of Self-adjoint Operators	179
8.6	Applications to Schrödinger Operators: Essential Spectrum	182
8.7	Exercises	185
9	Trace Class Perturbations of Spectra of Self-adjoint Operators	189
9.1	Parts of the Spectrum of a Self-adjoint Operator	189
9.2	Aronszajn–Donoghue Theory of Rank One Perturbations	194
9.3	Krein’s Spectral Shift for Rank One Perturbations	200
9.4	Infinite Determinants	203
9.5	Perturbation Determinants	206
9.6	Krein’s Spectral Shift for Trace Class Perturbations	209
9.7	Krein’s Trace Formula	213
9.8	Exercises	217
9.9	Notes to Part IV	218
Part V	Forms and Operators	
10	Semibounded Forms and Self-adjoint Operators	221
10.1	Closed and Closable Lower Semibounded Forms	221
10.2	Semibounded Closed Forms and Self-adjoint Operators	225
10.3	Order Relations for Self-adjoint Operators	230
10.4	The Friedrichs Extension of a Semibounded Symmetric Operator	233
10.5	Examples of Semibounded Forms and Operators	235
10.6	Dirichlet and Neumann Laplacians on Domains of $\mathbb{R}^d$	238
10.6.1	Laplacian with Dirichlet Boundary Condition	239
10.6.2	Laplacian with Neumann Boundary Condition	241
10.7	Perturbations of Forms and Form Sums	243
10.8	Exercises	248
11	Sectorial Forms and m-Sectorial Operators	251
11.1	Bounded Coercive Forms on Embedded Hilbert Spaces	251
11.2	Sectorial Forms	255
11.3	Application to Second-Order Elliptic Differential Operators	258
11.4	Exercises	262
12	Discrete Spectra of Self-adjoint Operators	265
12.1	The Min–Max Principle	265
12.2	Negative or Positive Eigenvalues of Schrödinger Operators	270
12.3	Asymptotic Distribution of Eigenvalues of the Dirichlet Laplacian	273
12.4	Exercises	278
12.5	Notes to Part V	280

Part VI Self-adjoint Extension Theory of Symmetric Operators

13 Self-adjoint Extensions: Cayley Transform and Krein Transform	283
13.1 The Cayley Transform of a Symmetric Operator	283
13.2 Von Neumann's Extension Theory of Symmetric Operators	287
13.3 Existence of Positive Self-adjoint Extensions and Krein-von Neumann Extensions	290
13.4 Positive Self-adjoint Extensions and Krein Transform	296
13.4.1 Self-adjoint Extensions of Bounded Symmetric Operators	296
13.4.2 The Krein Transform of a Positive Symmetric Operator	297
13.5 Self-adjoint Extensions Commuting with a Conjugation or Anticommuting with a Symmetry	299
13.6 Exercises	302
14 Self-adjoint Extensions: Boundary Triplets	307
14.1 Linear Relations	307
14.2 Boundary Triplets of Adjoints of Symmetric Operators	311
14.3 Operator-Theoretic Examples	315
14.4 Examples: Differentiation Operators III	318
14.5 Gamma Fields and Weyl Functions	322
14.6 The Krein-Naimark Resolvent Formula	325
14.7 Boundary Triplets and Semibounded Self-adjoint Operators	329
14.8 Positive Self-adjoint Extensions	333
14.9 Exercises	339
15 Sturm-Liouville Operators	343
15.1 Sturm-Liouville Operators with Regular End Points	343
15.2 Limit Circle Case and Limit Point Case	347
15.3 Boundary Triplets for Sturm-Liouville Operators	352
15.4 Resolvents of Self-adjoint Extensions	356
15.5 Exercises	361
16 The One-Dimensional Hamburger Moment Problem	363
16.1 Moment Problem and Jacobi Operators	363
16.2 Polynomials of Second Kind	374
16.3 The Indeterminate Hamburger Moment Problem	378
16.4 Nevanlinna Parameterization of Solutions	386
16.5 Exercises	389
16.6 Notes to Part VI	391
Appendix A Bounded Operators and Classes of Compact Operators on Hilbert Spaces	393
Appendix B Measure Theory	397
Appendix C The Fourier Transform	403
Appendix D Distributions and Sobolev Spaces	405
Appendix E Absolutely Continuous Functions	409

Appendix F	Nevanlinna Functions and Stieltjes Transforms	411
References		417
Classical Articles		417
Books		418
Articles		420
Subject Index		425
Symbol Index		431