## Contents

Preface to the third edition ..... XX
Preface to the second edition ..... xxiii
Preface to the first edition ..... XXV
1 Preliminary algebra ..... 1
1.1 Simple functions and equations ..... 1
Polynomial equations; factorisation; properties of roots
1.2 Trigonometric identities ..... 10
Single angle; compound angles; double- and half-angle identities
1.3 Coordinate geometry ..... 15
1.4 Partial fractions ..... 18Complications and special cases
1.5 Binomial expansion ..... 25
1.6 Properties of binomial coefficients ..... 27
1.7 Some particular methods of proof ..... 30
Proof by induction; proof by contradiction; necessary and sufficient conditions
1.8 Exercises ..... 36
1.9 Hints and answers ..... 39
2 Preliminary calculus ..... 41
2.1 Differentiation ..... 41
Differentiation from first principles; products; the chain rule; quotients; implicit differentiation; logarithmic differentiation; Leibnitz' theorem; special points of a function; curvature; theorems of differentiation
2.2 Integration ..... 59
Integration from first principles; the inverse of differentiation; by inspec- tion; sinusoidal functions; logarithmic integration; using partial fractions; substitution method; integration by parts; reduction formulae; infinite and improper integrals; plane polar coordinates; integral inequalities; applications of integration
2.3 Exercises ..... 76
2.4 Hints and answers ..... 81
3 Complex numbers and hyperbolic functions ..... 83
3.1 The need for complex numbers ..... 83
3.2 Manipulation of complex numbers ..... 85
Addition and subtraction; modulus and argument; multiplication; complex conjugate; division
3.3 Polar representation of complex numbers ..... 92
Multiplication and division in polar form
3.4 de Moivre's theorem ..... 95
trigonometric identities; finding the nth roots of unity; solving polynomial equations
3.5 Complex logarithms and complex powers ..... 99
3.6 Applications to differentiation and integration ..... 101
3.7 Hyperbolic functions ..... 102
Definitions; hyperbolic-trigonometric analogies; identities of hyperbolic functions; solving hyperbolic equations; inverses of hyperbolic functions; calculus of hyperbolic functions
3.8 Exercises ..... 109
3.9 Hints and answers ..... 113
4 Series and limits ..... 115
4.1 Series ..... 115
4.2 Summation of series ..... 116
Arithmetic series; geometric series; arithmetico-geometric series; the difference method; series involving natural numbers; transformation of series
4.3 Convergence of infinite series ..... 124
Absolute and conditional convergence; series containing only real positive terms; alternating series test
4.4 Operations with series ..... 131
4.5 Power series ..... 131
Convergence of power series; operations with power series
4.6 Taylor series ..... 136
Taylor's theorem; approximation errors; standard Maclaurin series
4.7 Evaluation of limits ..... 141
4.8 Exercises ..... 144
4.9 Hints and answers ..... 149
5 Partial differentiation ..... 151
5.1 Definition of the partial derivative ..... 151
5.2 The total differential and total derivative ..... 153
5.3 Exact and inexact differentials ..... 155
5.4 Useful theorems of partial differentiation ..... 157
5.5 The chain rule ..... 157
5.6 Change of variables ..... 158
5.7 Taylor's theorem for many-variable functions ..... 160
5.8 Stationary values of many-variable functions ..... 162
5.9 Stationary values under constraints ..... 167
5.10 Envelopes ..... 173
5.11 Thermodynamic relations ..... 176
5.12 Differentiation of integrals ..... 178
5.13 Exercises ..... 179
5.14 Hints and answers ..... 185
6 Multiple integrals ..... 187
6.1 Double integrals ..... 187
6.2 Triple integrals ..... 190
6.3 Applications of multiple integrals ..... 191
Areas and volumes; masses, centres of mass and centroids; Pappus' theorems; moments of inertia; mean values of functions
6.4 Change of variables in multiple integrals ..... 199
Change of variables in double integrals; evaluation of the integral $I=$ $\int_{-\infty}^{\infty} e^{-x^{2}} d x$; change of variables in triple integrals; general properties of Jacobians
6.5 Exercises ..... 207
6.6 Hints and answers ..... 211
$7 \quad$ Vector algebra ..... 212
7.1 Scalars and vectors ..... 212
7.2 Addition and subtraction of vectors ..... 213
7.3 Multiplication by a scalar ..... 214
7.4 Basis vectors and components ..... 217
7.5 Magnitude of a vector ..... 218
7.6 Multiplication of vectors ..... 219
Scalar product; vector product; scalar triple product; vector triple product
7.7 Equations of lines, planes and spheres ..... 226
7.8 Using vectors to find distances ..... 229
Point to line; point to plane; line to line; line to plane
7.9 Reciprocal vectors ..... 233
7.10 Exercises ..... 234
7.11 Hints and answers ..... 240
8 Matrices and vector spaces ..... 241
8.1 Vector spaces ..... 242
Basis vectors; inner product; some useful inequalities
8.2 Linear operators ..... 247
8.3 Matrices ..... 249
8.4 Basic matrix algebra ..... 250Matrix addition; multiplication by a scalar; matrix multiplication
8.5 Functions of matrices ..... 255
8.6 The transpose of a matrix ..... 255
8.7 The complex and Hermitian conjugates of a matrix ..... 256
8.8 The trace of a matrix ..... 258
8.9 The determinant of a matrix ..... 259Properties of determinants
8.10 The inverse of a matrix ..... 263
8.11 The rank of a matrix ..... 267
8.12 Special types of square matrix ..... 268
Diagonal; triangular; symmetric and antisymmetric; orthogonal; Hermitian and anti-Hermitian; unitary; normal
8.13 Eigenvectors and eigenvalues ..... 272
Of a normal matrix; of Hermitian and anti-Hermitian matrices; of a unitary matrix; of a general square matrix
8.14 Determination of eigenvalues and eigenvectors ..... 280 Degenerate eigenvalues
8.15 Change of basis and similarity transformations ..... 282
8.16 Diagonalisation of matrices ..... 285
8.17 Quadratic and Hermitian forms ..... 288Stationary properties of the eigenvectors; quadratic surfaces
8.18 Simultaneous linear equations ..... 292
Range; null space; $N$ simultaneous linear equations in $N$ unknowns; singular value decomposition
8.19 Exercises ..... 307
8.20 Hints and answers ..... 314
9 Normal modes ..... 316
9.1 Typical oscillatory systems ..... 317
9.2 Symmetry and normal modes ..... 322
9.3 Rayleigh-Ritz method ..... 327
9.4 Exercises ..... 329
9.5 Hints and answers ..... 332
10 Vector calculus ..... 334
10.1 Differentiation of vectors ..... 334
Composite vector expressions; differential of a vector
10.2 Integration of vectors ..... 339
10.3 Space curves ..... 340
10.4 Vector functions of several arguments ..... 344
10.5 Surfaces ..... 345
10.6 Scalar and vector fields ..... 347
10.7 Vector operators ..... 347
Gradient of a scalar field; divergence of a vector field; curl of a vector field
10.8 Vector operator formulae ..... 354
Vector operators acting on sums and products; combinations of grad, div and curl
10.9 Cylindrical and spherical polar coordinates ..... 357
10.10 General curvilinear coordinates ..... 364
10.11 Exercises ..... 369
10.12 Hints and answers ..... 375
11 Line, surface and volume integrals ..... 377
11.1 Line integrals ..... 377
Evaluating line integrals; physical examples; line integrals with respect to a scalar
11.2 Connectivity of regions ..... 383
11.3 Green's theorem in a plane ..... 384
11.4 Conservative fields and potentials ..... 387
11.5 Surface integrals ..... 389Evaluating surface integrals; vector areas of surfaces; physical examples
11.6 Volume integrals ..... 396Volumes of three-dimensional regions
11.7 Integral forms for grad, div and curl ..... 398
11.8 Divergence theorem and related theorems ..... 401
Green's theorems; other related integral theorems; physical applications
11.9 Stokes' theorem and related theorems ..... 406
Related integral theorems; physical applications
11.10 Exercises ..... 409
11.11 Hints and answers ..... 414
12 Fourier series ..... 415
12.1 The Dirichlet conditions ..... 415
12.2 The Fourier coefficients ..... 417
12.3 Symmetry considerations ..... 419
12.4 Discontinuous functions ..... 420
12.5 Non-periodic functions ..... 422
12.6 Integration and differentiation ..... 424
12.7 Complex Fourier series ..... 424
12.8 Parseval's theorem ..... 426
12.9 Exercises ..... 427
12.10 Hints and answers ..... 431
13 Integral transforms ..... 433
13.1 Fourier transforms ..... 433
The uncertainty principle; Fraunhofer diffraction; the Dirac $\delta$-function; relation of the $\delta$-function to Fourier transforms; properties of Fourier transforms; odd and even functions; convolution and deconvolution; correlation functions and energy spectra; Parseval's theorem; Fourier transforms in higher dimensions
13.2 Laplace transforms ..... 453
Laplace transforms of derivatives and integrals; other properties of Laplace transforms
13.3 Concluding remarks ..... 459
13.4 Exercises ..... 460
13.5 Hints and answers ..... 466
14 First-order ordinary differential equations ..... 468
14.1 General form of solution ..... 469
14.2 First-degree first-order equations ..... 470
Separable-variable equations; exact equations; inexact equations, integrat- ing factors; linear equations; homogeneous equations; isobaric equations; Bernoulli's equation; miscellaneous equations
14.3 Higher-degree first-order equations ..... 480
Equations soluble for p; for $x$; for $y$; Clairaut's equation
14.4 Exercises ..... 484
14.5 Hints and answers ..... 488
15 Higher-order ordinary differential equations ..... 490
15.1 Linear equations with constant coefficients ..... 492
Finding the complementary function $y_{\mathrm{c}}(x)$; finding the particular integral $y_{\mathrm{p}}(x)$; constructing the general solution $y_{\mathrm{c}}(x)+y_{\mathrm{p}}(x)$; linear recurrence relations; Laplace transform method
15.2 Linear equations with variable coefficients ..... 503
The Legendre and Euler linear equations; exact equations; partially known complementary function; variation of parameters; Green's functions; canonical form for second-order equations
15.3 General ordinary differential equations ..... 518
Dependent variable absent; independent variable absent; non-linear exact equations; isobaric or homogeneous equations; equations homogeneous in $x$ or $y$ alone; equations having $y=A e^{x}$ as a solution
15.4 Exercises ..... 523
15.5 Hints and answers ..... 529
16 Series solutions of ordinary differential equations ..... 531
16.1 Second-order linear ordinary differential equations ..... 531
Ordinary and singular points
16.2 Series solutions about an ordinary point ..... 535
16.3 Series solutions about a regular singular point ..... 538
Distinct roots not differing by an integer; repeated root of the indicial equation; distinct roots differing by an integer
16.4 Obtaining a second solution ..... 544
The Wronskian method; the derivative method; series form of the second solution
16.5 Polynomial solutions ..... 548
16.6 Exercises ..... 550
16.7 Hints and answers ..... 553
17 Eigenfunction methods for differential equations ..... 554
17.1 Sets of functions ..... 556Some useful inequalities
17.2 Adjoint, self-adjoint and Hermitian operators ..... 559
17.3 Properties of Hermitian operators ..... 561
Reality of the eigenvalues; orthogonality of the eigenfunctions; construction of real eigenfunctions
17.4 Sturm-Liouville equations ..... 564
Valid boundary conditions; putting an equation into Sturm-Liouville form
17.5 Superposition of eigenfunctions: Green's functions ..... 569
17.6 A useful generalisation ..... 572
17.7 Exercises ..... 573
17.8 Hints and answers ..... 576
18 Special functions ..... 577
18.1 Legendre functions ..... 577General solution for integer $\ell$; properties of Legendre polynomials
18.2 Associated Legendre functions ..... 587
18.3 Spherical harmonics ..... 593
18.4 Chebyshev functions ..... 595
18.5 Bessel functions ..... 602
General solution for non-integer $v$; general solution for integer $v$; properties of Bessel functions
18.6 Spherical Bessel functions ..... 614
18.7 Laguerre functions ..... 616
18.8 Associated Laguerre functions ..... 621
18.9 Hermite functions ..... 624
18.10 Hypergeometric functions ..... 628
18.11 Confluent hypergeometric functions ..... 633
18.12 The gamma function and related functions ..... 635
18.13 Exercises ..... 640
18.14 Hints and answers ..... 646
19 Quantum operators ..... 648
19.1 Operator formalism ..... 648Commutators
19.2 Physical examples of operators ..... 656
Uncertainty principle; angular momentum; creation and annihilation operators
19.3 Exercises ..... 671
19.4 Hints and answers ..... 674
20 Partial differential equations: general and particular solutions ..... 675
20.1 Important partial differential equations ..... 676
The wave equation; the diffusion equation; Laplace's equation; Poisson's equation; Schrödinger's equation
20.2 General form of solution ..... 680
20.3 General and particular solutions ..... 681
First-order equations; inhomogeneous equations and problems; second-order equations
20.4 The wave equation ..... 693
20.5 The diffusion equation ..... 695
20.6 Characteristics and the existence of solutions ..... 699
First-order equations; second-order equations
20.7 Uniqueness of solutions ..... 705
20.8 Exercises ..... 707
20.9 Hints and answers ..... 711
21 Partial differential equations: separation of variables and other methods ..... 713
21.1 Separation of variables: the general method ..... 713
21.2 Superposition of separated solutions ..... 717
21.3 Separation of variables in polar coordinates ..... 725
Laplace's equation in polar coordinates; spherical harmonics; other equations in polar coordinates; solution by expansion; separation of variables for inhomogeneous equations
21.4 Integral transform methods ..... 747
21.5 Inhomogeneous problems - Green's functions ..... 751
Similarities to Green's functions for ordinary differential equations; general boundary-value problems; Dirichlet problems; Neumann problems
21.6 Exercises ..... 767
21.7 Hints and answers ..... 773
22 Calculus of variations ..... 775
22.1 The Euler-Lagrange equation ..... 776
22.2 Special cases ..... 777
$F$ does not contain $y$ explicitly; $F$ does not contain $x$ explicitly
22.3 Some extensions ..... 781
Several dependent variables; several independent variables; higher-order derivatives; variable end-points
22.4 Constrained variation ..... 785
22.5 Physical variational principles ..... 787
Fermat's principle in optics; Hamilton's principle in mechanics
22.6 General eigenvalue problems ..... 790
22.7 Estimation of eigenvalues and eigenfunctions ..... 792
22.8 Adjustment of parameters ..... 795
22.9 Exercises ..... 797
22.10 Hints and answers ..... 801
23 Integral equations ..... 803
23.1 Obtaining an integral equation from a differential equation ..... 803
23.2 Types of integral equation ..... 804
23.3 Operator notation and the existence of solutions ..... 805
23.4 Closed-form solutions ..... 806
Separable kernels; integral transform methods; differentiation
23.5 Neumann series ..... 813
23.6 Fredholm theory ..... 815
23.7 Schmidt-Hilbert theory ..... 816
23.8 Exercises ..... 819
23.9 Hints and answers ..... 823
24 Complex variables ..... 824
24.1 Functions of a complex variable ..... 825
24.2 The Cauchy-Riemann relations ..... 827
24.3 Power series in a complex variable ..... 830
24.4 Some elementary functions ..... 832
24.5 Multivalued functions and branch cuts ..... 835
24.6 Singularities and zeros of complex functions ..... 837
24.7 Conformal transformations ..... 839
24.8 Complex integrals ..... 845
24.9 Cauchy's theorem ..... 849
24.10 Cauchy's integral formula ..... 851
24.11 Taylor and Laurent series ..... 853
24.12 Residue theorem ..... 858
24.13 Definite integrals using contour integration ..... 861
24.14 Exercises ..... 867
24.15 Hints and answers ..... 870
25 Applications of complex variables ..... 871
25.1 Complex potentials ..... 871
25.2 Applications of conformal transformations ..... 876
25.3 Location of zeros ..... 879
25.4 Summation of series ..... 882
25.5 Inverse Laplace transform ..... 884
25.6 Stokes' equation and Airy integrals ..... 888
25.7 WKB methods ..... 895
25.8 Approximations to integrals ..... 905
Level lines and saddle points; steepest descents; stationary phase
25.9 Exercises ..... 920
25.10 Hints and answers ..... 925
26 Tensors ..... 927
26.1 Some notation ..... 928
26.2 Change of basis ..... 929
26.3 Cartesian tensors ..... 930
26.4 First- and zero-order Cartesian tensors ..... 932
26.5 Second- and higher-order Cartesian tensors ..... 935
26.6 The algebra of tensors ..... 938
26.7 The quotient law ..... 939
26.8 The tensors $\delta_{i j}$ and $\epsilon_{i j k}$ ..... 941
26.9 Isotropic tensors ..... 944
26.10 Improper rotations and pseudotensors ..... 946
26.11 Dual tensors ..... 949
26.12 Physical applications of tensors ..... 950
26.13 Integral theorems for tensors ..... 954
26.14 Non-Cartesian coordinates ..... 955
26.15 The metric tensor ..... 957
26.16 General coordinate transformations and tensors ..... 960
26.17 Relative tensors ..... 963
26.18 Derivatives of basis vectors and Christoffel symbols ..... 965
26.19 Covariant differentiation ..... 968
26.20 Vector operators in tensor form ..... 971
26.21 Absolute derivatives along curves ..... 975
26.22 Geodesics ..... 976
26.23 Exercises ..... 977
26.24 Hints and answers ..... 982
27 Numerical methods ..... 984
27.1 Algebraic and transcendental equations ..... 985
Rearrangement of the equation; linear interpolation; binary chopping; Newton-Raphson method
27.2 Convergence of iteration schemes ..... 992
27.3 Simultaneous linear equations ..... 994
Gaussian elimination; Gauss-Seidel iteration; tridiagonal matrices
27.4 Numerical integration ..... 1000
Trapezium rule; Simpson's rule; Gaussian integration; Monte Carlo methods
27.5 Finite differences ..... 1019
27.6 Differential equations ..... 1020
Difference equations; Taylor series solutions; prediction and correction; Runge-Kutta methods; isoclines
27.7 Higher-order equations ..... 1028
27.8 Partial differential equations ..... 1030
27.9 Exercises ..... 1033
27.10 Hints and answers ..... 1039
28 Group theory ..... 1041
28.1 Groups ..... 1041
Definition of a group; examples of groups
28.2 Finite groups ..... 1049
28.3 Non-Abelian groups ..... 1052
28.4 Permutation groups ..... 1056
28.5 Mappings between groups ..... 1059
28.6 Subgroups ..... 1061
28.7 Subdividing a group ..... 1063
Equivalence relations and classes; congruence and cosets; conjugates and classes
28.8 Exercises ..... 1070
28.9 Hints and answers ..... 1074
29 Representation theory ..... 1076
29.1 Dipole moments of molecules ..... 1077
29.2 Choosing an appropriate formalism ..... 1078
29.3 Equivalent representations ..... 1084
29.4 Reducibility of a representation ..... 1086
29.5 The orthogonality theorem for irreducible representations ..... 1090
29.6 Characters ..... 1092
Orthogonality property of characters
29.7 Counting irreps using characters ..... 1095
Summation rules for irreps
29.8 Construction of a character table ..... 1100
29.9 Group nomenclature ..... 1102
29.10 Product representations ..... 1103
29.11 Physical applications of group theory ..... 1105
Bonding in molecules; matrix elements in quantum mechanics; degeneracy of normal modes; breaking of degeneracies
29.12 Exercises ..... 1113
29.13 Hints and answers ..... 1117
30 Probability ..... 1119
30.1 Venn diagrams ..... 1119
30.2 Probability ..... 1124
Axioms and theorems; conditional probability; Bayes' theorem
30.3 Permutations and combinations ..... 1133
30.4 Random variables and distributions ..... 1139
Discrete random variables; continuous random variables
30.5 Properties of distributions ..... 1143
Mean; mode and median; variance and standard deviation; moments; central moments
30.6 Functions of random variables ..... 1150
30.7 Generating functions ..... 1157
Probability generating functions; moment generating functions; characteristic functions; cumulant generating functions
30.8 Important discrete distributions ..... 1168
Binomial; geometric; negative binomial; hypergeometric; Poisson
30.9 Important continuous distributions ..... 1179
Gaussian; log-normal; exponential; gamma; chi-squared; Cauchy; Breit- Wigner; uniform
30.10 The central limit theorem ..... 1195
30.11 Joint distributions ..... 1196
Discrete bivariate; continuous bivariate; marginal and conditional distributions
30.12 Properties of joint distributions ..... 1199 Means; variances; covariance and correlation
30.13 Generating functions for joint distributions ..... 1205
30.14 Transformation of variables in joint distributions ..... 1206
30.15 Important joint distributions ..... 1207
Multinominal; multivariate Gaussian
30.16 Exercises ..... 1211
30.17 Hints and answers ..... 1219
31 Statistics ..... 1221
31.1 Experiments, samples and populations ..... 1221
31.2 Sample statistics ..... 1222Averages; variance and standard deviation; moments; covariance and correlation
31.3 Estimators and sampling distributions ..... 1229
Consistency, bias and efficiency; Fisher's inequality; standard errors; confi- dence limits
31.4 Some basic estimators ..... 1243
Mean; variance; standard deviation; moments; covariance and correlation
31.5 Maximum-likelihood method ..... 1255
ML estimator; transformation invariance and bias; efficiency; errors and confidence limits; Bayesian interpretation; large- $N$ behaviour; extended ML method
31.6 The method of least squares ..... 1271
Linear least squares; non-linear least squares
31.7 Hypothesis testing ..... 1277
Simple and composite hypotheses; statistical tests; Neyman-Pearson; gener- alised likelihood-ratio; Student's $t$; Fisher's F; goodness of fit
31.8 Exercises ..... 1298
31.9 Hints and answers ..... 1303
Index ..... 1305

