

Contents

1	Fluid Flow Modeling	1
1.1	Fluids in Continuum Mechanics	2
1.2	Balance Laws	4
1.3	Field Equations	8
1.3.1	Conservation of Mass	9
1.3.2	Balance of Linear Momentum	10
1.3.3	Total Energy	12
1.3.4	Entropy	13
1.4	Constitutive Relations	14
1.4.1	Molecular Energy and Transport Terms	14
1.4.2	State Equations	15
1.4.3	Effect of Thermal Radiation	17
1.4.4	Typical Values of Some Physical Coefficients	18
2	Weak Solutions, A Priori Estimates	21
2.1	Weak Formulation	23
2.1.1	Equation of Continuity	24
2.1.2	Balance of Linear Momentum	24
2.1.3	Balance of Total Energy	26
2.1.4	Entropy Production	26
2.1.5	Constitutive Relations	27
2.2	A Priori Estimates	28
2.2.1	Total Mass Conservation	28
2.2.2	Energy Estimates	28
2.2.3	Estimates Based on Second Law of Thermodynamics	30
2.2.4	Positivity of the Absolute Temperature	36
2.2.5	Pressure Estimates	39
2.2.6	Pressure Estimates, an Alternative Approach	44

3 Existence Theory	49
3.1 Hypotheses	50
3.2 Structural Properties of Constitutive Functions	53
3.3 Main Existence Result	57
3.3.1 Approximation Scheme	58
3.4 Solvability of the Approximate System.....	60
3.4.1 Approximate Continuity Equation	61
3.4.2 Approximate Internal Energy Equation	63
3.4.3 Local Solvability of the Approximate Problem	71
3.4.4 Uniform Estimates and Global Existence	73
3.5 Faedo-Galerkin Limit.....	78
3.5.1 Estimates Independent of the Dimension of Faedo-Galerkin Approximations	79
3.5.2 Limit Passage in the Approximate Continuity Equation	82
3.5.3 Strong Convergence of the Approximate Temperatures and the Limit in the Entropy Equation	85
3.5.4 Limit in the Approximate Momentum Equation.....	93
3.5.5 The Limit System Resulting from the Faedo-Galerkin Approximation	94
3.5.6 The Entropy Production Rate Represented by a Positive Measure	96
3.6 Artificial Diffusion Limit	97
3.6.1 Uniform Estimates and Limit in the Approximate Continuity Equation	97
3.6.2 Entropy Balance and Strong Convergence of the Approximate Temperatures	100
3.6.3 Uniform Pressure Estimates	106
3.6.4 Limit in the Approximate Momentum Equation and in the Energy Balance	108
3.6.5 Strong Convergence of the Densities	109
3.6.6 Artificial Diffusion Asymptotic Limit.....	118
3.7 Vanishing Artificial Pressure	119
3.7.1 Uniform Estimates	120
3.7.2 Asymptotic Limit for Vanishing Artificial Pressure ...	122
3.7.3 Entropy Balance and Pointwise Convergence of the Temperature	125
3.7.4 Pointwise Convergence of the Densities	129
3.7.5 Oscillations Defect Measure	135
3.8 Regularity Properties of the Weak Solutions	140
4 Asymptotic Analysis: An Introduction	145
4.1 Scaling and Scaled Equations	147
4.2 Low Mach Number Limits	149

4.3	Strongly Stratified Flows	151
4.4	Acoustic Waves	154
4.4.1	Low Stratification	154
4.4.2	Strong Stratification	156
4.4.3	Attenuation of Acoustic Waves	156
4.5	Acoustic Analogies	158
4.6	Initial Data	160
4.7	A General Approach to Singular Limits for the Full Navier-Stokes-Fourier System	161
5	Singular Limits: Low Stratification	167
5.1	Hypotheses and Global Existence for the Primitive System	170
5.1.1	Hypotheses	171
5.1.2	Global-in-Time Solutions	172
5.2	Dissipation Equation, Uniform Estimates	174
5.2.1	Conservation of Total Mass	174
5.2.2	Total Dissipation Balance and Related Estimates	175
5.2.3	Uniform Estimates	179
5.3	Convergence	182
5.3.1	Equation of Continuity	184
5.3.2	Entropy Balance	185
5.3.3	Momentum Equation	188
5.4	Convergence of the Convective Term	192
5.4.1	Helmholtz Decomposition	193
5.4.2	Compactness of the Solenoidal Part	194
5.4.3	Acoustic Equation	196
5.4.4	Formal Analysis of Acoustic Equation	199
5.4.5	Spectral Analysis of the Wave Operator	202
5.4.6	Reduction to a Finite Number of Modes	203
5.4.7	Weak Limit of the Convective Term: Time Lifting	205
5.5	Conclusion: Main Result	208
5.5.1	Weak Formulation of the Target Problem	208
5.5.2	Main Result	210
5.5.3	Determining the Initial Temperature Distribution	211
5.5.4	Energy Inequality for the Limit System	212
6	Stratified Fluids	221
6.1	Motivation	221
6.2	Primitive System	222
6.2.1	Field Equations	222
6.2.2	Constitutive Relations	223
6.2.3	Scaling	225
6.3	Asymptotic Limit	227
6.3.1	Static States	227
6.3.2	Solutions to the Primitive System	228
6.3.3	Main Result	230

6.4	Uniform Estimates	233
6.4.1	Dissipation Equation, Energy Estimates	234
6.4.2	Pressure Estimates	240
6.5	Convergence Towards the Target System	242
6.5.1	Anelastic Constraint.....	243
6.5.2	Determining the Pressure	243
6.5.3	Driving Force	246
6.5.4	Momentum Equation.....	248
6.6	Analysis of the Acoustic Waves.....	249
6.6.1	Acoustic Equation	250
6.6.2	Spectral Analysis of the Wave Operator	252
6.6.3	Convergence of the Convective Term	254
6.7	Asymptotic Limit in the Entropy Balance	259
7	Interaction of Acoustic Waves with Boundary	263
7.1	Problem Formulation	266
7.1.1	Field Equations	266
7.1.2	Physical Domain and Boundary Conditions	267
7.2	Main Result: The No-Slip Boundary Conditions	269
7.2.1	Preliminaries: Global Existence	270
7.2.2	Compactness of the Family of Velocities	272
7.3	Uniform Estimates	273
7.4	Analysis of Acoustic Waves	275
7.4.1	Acoustic Equation	275
7.4.2	Spectral Analysis of the Acoustic Operator.....	278
7.5	Strong Convergence of the Velocity Field	288
7.5.1	Compactness of the Solenoidal Component	289
7.5.2	Reduction to a Finite Number of Modes	290
7.5.3	Strong Convergence	291
7.6	Asymptotic Limit on Domains with Oscillatory Boundaries and Complete Slip Boundary Conditions.....	297
7.7	Uniform Bounds	300
7.8	Convergence of the Velocity Trace on Oscillatory Boundary...	301
7.9	Strong Convergence of the Velocity Field Revisited	304
7.9.1	Solenoidal Component	304
7.9.2	Acoustic Waves.....	305
7.9.3	Strong Convergence of the Gradient Component.....	308
7.10	Concluding Remarks.....	311
8	Problems on Large Domains	313
8.1	Primitive System	313
8.2	Oberbeck–Boussinesq Approximation in Exterior Domains ...	317
8.3	Uniform Estimates	318
8.3.1	Static Solutions	318
8.3.2	Estimates Based on the Hypothesis of Thermodynamic Stability	319

8.3.3	Estimates Based on the Specific Form of Constitutive Relations	322
8.4	Convergence, Part I	325
8.5	Acoustic Equation	326
8.5.1	Boundedness of the Data.....	329
8.5.2	Acoustic Equation Revisited.....	333
8.6	Regularization and Extension to Ω	335
8.6.1	Regularization	335
8.6.2	Reduction to Smooth Data	338
8.7	Dispersive Estimates and Time Decay of Acoustic Waves	349
8.7.1	Compactness of the Solenoidal Components	350
8.7.2	Analysis of Acoustic Waves	350
8.7.3	Decay Estimates via RAGE Theorem	355
8.8	Convergence to the Target System	358
8.9	Dispersive Estimates Revisited	361
8.9.1	RAGE Theorem via Spectral Measures	361
8.9.2	Decay Estimates via Kato's Theorem	363
8.10	Conclusion	367
9	Vanishing Dissipation Limits	369
9.1	Problem Formulation	370
9.1.1	Physical Space and Boundary Conditions	371
9.1.2	Initial Data	372
9.1.3	Target Problem	372
9.1.4	Strategy of the Proof of Stability of Smooth Solutions to the Target Problem	374
9.2	Relative Energy Inequality	374
9.3	Uniform Estimates	378
9.4	Well-Prepared Initial Data	381
9.5	Ill-Prepared Initial Data.....	389
9.5.1	Acoustic Equation	389
9.5.2	Transport Equation	391
9.5.3	Stability via the Relative Energy Inequality	392
9.5.4	Conclusion	406
10	Acoustic Analogies.....	409
10.1	Asymptotic Analysis and the Limit System	410
10.2	Acoustic Equation Revisited	412
10.3	Two-Scale Convergence	416
10.3.1	Approximate Methods	421
10.4	Lighthill's Acoustic Analogy in the Low Mach Number Regime	422
10.4.1	Ill-Prepared Data	422
10.4.2	Well-Prepared Data	423
10.5	Concluding Remarks.....	426

11 Appendix	429
11.1 Spectral Theory of Self-Adjoint Operators	429
11.2 Mollifiers	432
11.3 Basic Properties of Some Elliptic Operators	433
11.3.1 A Priori Estimates	434
11.3.2 Fredholm Alternative	437
11.3.3 Spectrum of a Generalized Laplacian	438
11.3.4 Neumann Laplacian on Unbounded Domains	440
11.4 Normal Traces.....	443
11.5 Singular and Weakly Singular Operators	447
11.6 The Inverse of the div -Operator (Bogovskii Formula)	448
11.7 Helmholtz Decomposition	458
11.8 Function Spaces of Hydrodynamics	460
11.9 Poincaré Type Inequalities	462
11.10 Korn Type Inequalities.....	464
11.11 Estimating $\nabla \mathbf{u}$ by Means of $\operatorname{div}_x \mathbf{u}$ and $\operatorname{curl}_x \mathbf{u}$	469
11.12 Weak Convergence and Monotone Functions	471
11.13 Weak Convergence and Convex Functions	476
11.14 Div-Curl Lemma	480
11.15 Maximal Regularity for Parabolic Equations	482
11.16 Quasilinear Parabolic Equations	484
11.17 Basic Properties of the Riesz Transform and Related Operators	486
11.18 Commutators Involving Riesz Operators	489
11.19 Renormalized Solutions to the Equation of Continuity	492
11.20 Transport Equation and the Euler System	499
12 Bibliographical Remarks	501
12.1 Fluid Flow Modeling	501
12.2 Mathematical Theory of Weak Solutions	502
12.3 Existence Theory.....	503
12.4 Analysis of Singular Limits	503
12.5 Propagation of Acoustic Waves	504
12.6 Relative Energy, Inviscid Limits	505
Bibliography	507
Index	519