

Dietmar A. Salamon

## Measure and Integration

The book is intended as a companion to a one semester introductory lecture course on measure and integration. After an introduction to abstract measure theory it proceeds to the construction of the Lebesgue measure and of Borel measures on locally compact Hausdorff spaces,  $L^p$  spaces and their dual spaces and elementary Hilbert space theory. Special features include the formulation of the Riesz Representation Theorem in terms of both inner and outer regularity, the proofs of the Marcinkiewicz Interpolation Theorem and the Calderon–Zygmund inequality as applications of Fubini’s theorem and Lebesgue differentiation, the treatment of the generalized Radon–Nikodym theorem due to Fremlin, and the existence proof for Haar measures. Three appendices deal with Urysohn’s Lemma, product topologies, and the inverse function theorem.

The book assumes familiarity with first year analysis and linear algebra. It is suitable for second year undergraduate students of mathematics or anyone desiring an introduction to the concepts of measure and integration.

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