

"This book succeeds in unifying the field by bringing in disparate topics, already available in several papers, but not easy to understand, under one roof. ... it exceeds all expectations, in particular regarding the extent to which complex differential geometric notions permeate statistics."

—From the Foreword by Victor Pambuccian, Professor of Mathematics, Arizona State University

Nonparametric Statistics on Manifolds and Their Applications to Object Data Analysis provides one of the first thorough treatments of the theory and methodology for analyzing data on manifolds. It also presents in-depth applications to practical problems arising in a variety of fields, including statistics, medical imaging, computer vision, pattern recognition, and bioinformatics.

The book begins with a survey of illustrative examples of object data before moving to a review of concepts from mathematical statistics, differential geometry, and topology. The authors next describe theory and methods for working on various manifolds, giving a historical perspective of concepts from mathematics and statistics. They then present problems from a wide variety of areas, including diffusion tensor imaging, similarity shape analysis, directional data analysis, and projective shape analysis for machine vision. The book concludes with a discussion of current related research and graduate-level teaching topics as well as considerations related to computational statistics.

Features

- Introduces the relevant background material, providing you with the required concepts from statistics and differential geometry
- Presents detailed applications to various types of object data that illustrate where and how the theory and methods can be used in many fields
- Reinforces the underlying theoretical concepts with numerous exercises that encourage you to think more in depth



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