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## Institute of Mathematical Statistics Monographs

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This book introduces in a systematic manner a general nonparametric theory of statistics on manifolds, with emphasis on manifolds of shapes. The theory has important and varied applications in medical diagnostics, image analysis, and machine vision. An early chapter of examples establishes the effectiveness of the new methods and demonstrates how they outperform their parametric counterparts.

Inference is developed for both intrinsic and extrinsic Fréchet means of probability distributions on manifolds, and then applied to spaces of shapes defined as orbits of landmarks under a Lie group of transformations – in particular, similarity, reflection similarity, affine, and projective transformations. In addition, nonparametric Bayesian theory is adapted and extended to manifolds for the purposes of density estimation, regression, and classification. Ideal for statisticians who analyze manifold data and wish to develop their own methodology, this book is also of interest to probabilists, mathematicians, computer scientists, and morphometricians with mathematical training.



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