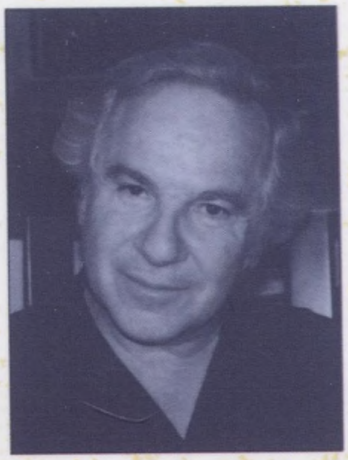

Michael Rosen

Number Theory in Function Fields

Elementary number theory is concerned with arithmetic properties of the ring of integers. Early in the development of number theory, it was noticed that the ring of integers has many properties in common with the ring of polynomials over a finite field. The first part of this book illustrates this relationship by presenting, for example, analogues of the theorems of Fermat and Euler, Wilson's theorem, quadratic (and higher) reciprocity, the prime number theorem, and Dirichlet's theorem on primes in an arithmetic progression. After presenting the required foundational material on function fields, the later chapters explore the analogy between global function fields and algebraic number fields. A variety of topics are presented, including the ABC-conjecture, Artin's conjecture on primitive roots, the Brumer–Stark conjecture, Drinfeld modules, class number formulae, and average value theorems.

The first few chapters of this book are accessible to advanced undergraduates. The later chapters are designed for graduate students and professionals in mathematics and related fields who want to learn more about the very fruitful relationship between number theory in algebraic number fields and algebraic function fields. In this book many paths are set forth for future learning and exploration.

Michael Rosen is Professor of Mathematics at Brown University, where he has been since 1962. He has published over 40 research papers, and he is the coauthor of *A Classical Introduction to Modern Number Theory*, with Kenneth Ireland. He received the Chauvenet Prize of the Mathematical Association of America in 1999 and the Philip J. Bray Teaching Award of Brown University in 2001.

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Preface vii

1 Polynomials over Finite Fields 1
Exercises 7

2 Primes, Arithmetic Functions, and the Zeta Function 11
Exercises 19

3 The Reciprocity Law 23
Exercises 30

4 Dirichlet L-Series and Primes in an Arithmetic Progression 33
Exercises 43

5 Algebraic Function Fields and Global Function Fields 45
Exercises 59

6 Weil Differentials and the Canonical Class 63
Exercises 75

7 Extensions of Function Fields, Riemann-Hurwitz,
and the ABC Theorem 77
Exercises 98

8 Constant Field Extensions 101
Exercises 112

9 Galois Extensions - Hecke and Artin L-Series 115
Exercises 145

10 Artin's Primitive Root Conjecture 149
Exercises 166

11 The Behavior of the Class Group in Constant Field Extensions 169
Exercises 190

12 Cyclotomic Function Fields 193
Exercises 216

13 Drinfeld Modules: An Introduction 219
Exercises 239

14 S-Units, S-Class Group, and the Corresponding L-Functions 241
Exercises 256

15 The Brumer-Stark Conjecture 257
Exercises 278

16 The Class Number Formulas in Quadratic and Cyclotomic Function Fields 283
Exercises 302

17 Average Value Theorems in Function Fields 305
Exercises 326

Appendix: A Proof of the Function Field Riemann Hypothesis 329

Bibliography 341

Author Index 353

Subject Index 355