

Probability and Its Applications

Olav Kallenberg

Foundations of Modern Probability

Second Edition

From the reviews of the first edition:

"... Kallenberg's present book would have to qualify as the assimilation of probability par excellence. It is a great edifice of material, clearly and ingeniously presented, without any non-mathematical distractions. Readers wishing to venture into it may do so with confidence that they are in very capable hands."

—F.B. Knight, *Mathematical Reviews*

"... Indeed the monograph has the potential to become a (possibly even "the") major reference book on large parts of probability theory for the next decade or more."

—M. Scheutzw, *Zentralblatt*

"The theory of probability has grown exponentially during the second half of the twentieth century and the idea of writing a single volume that could serve as a general reference for much of the modern theory seems almost foolhardy. Yet this is precisely what Professor Kallenberg has attempted in the volume under review and he has accomplished it brilliantly.... It is astonishing that a single volume of just over five hundred pages could contain so much material presented with complete rigor and still be at least formally self-contained...."

—R.K. Getoor, *Metrika*

This new edition contains four new chapters as well as numerous improvements throughout the text.

Olav Kallenberg was educated in Sweden, where he received his Ph.D. in 1972 from Chalmers University. After teaching for many years at Swedish universities, he moved in 1985 to the United States, where he is currently Professor of Mathematics at Auburn University. He is known for his book *Random Measures* (fourth edition, 1986) and for numerous research papers in all areas of probability. In 1977, he was the second recipient ever of the prestigious Rollo Davidson Prize from Cambridge University. In 1991 to 1994, he served as the Editor-in-Chief of *Probability Theory and Related Fields*.



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