Contents

CHAPTER 1. PROBABILITY 1

1. Borel's Normal Number Theorem, 1

*The Unit Interval—The Weak Law of Large Numbers—The Strong Law of Large Numbers—Strong Law Versus Weak—Length—The Measure Theory of Diophantine Approximation**

2. Probability Measures, 17

Spaces—Assigning Probabilities—Classes of Sets—Probability Measures—Lebesgue Measure on the Unit Interval—Sequence Space—Constructing σ-Fields**

3. Existence and Extension, 36

*Construction of the Extension—Uniqueness and the π-λ Theorem—Monotone Classes—Lebesgue Measure on the Unit Interval—Completeness—Nonmeasurable Sets—Two Impossibility Theorems**

4. Denumerable Probabilities, 51

General Formulas—Limit Sets—Independent Events—Subfields—The Borel–Cantelli Lemmas—The Zero–One Law

5. Simple Random Variables, 67

Definition—Convergence of Random Variables—Independence—Existence of Independent Sequences—Expected Value—Inequalities

6. The Law of Large Numbers, 85

The Strong Law—The Weak Law—Bernstein's Theorem—A Refinement of the Second Borel–Cantelli Lemma

*Stars indicate topics that may be omitted on a first reading.

7. Gambling Systems, 92

Gambler's Ruin—Selection Systems—Gambling Policies—Bold Play—Timid Play**

8. Markov Chains, 111

Definitions—Higher-Order Transitions—An Existence Theorem—Transience and Persistence—Another Criterion for Persistence—Stationary Distributions—Exponential Convergence—Optimal Stopping**

9. Large Deviations and the Law of the Iterated Logarithm,* 145

Moment Generating Functions—Large Deviations—Chernoff's Theorem—The Law of the Iterated Logarithm

CHAPTER 2. MEASURE 158

10. General Measures, 158

Classes of Sets—Conventions Involving ∞—Measures—Uniqueness

11. Outer Measure, 165

Outer Measure—Extension—An Approximation Theorem

12. Measures in Euclidean Space, 171

*Lebesgue Measure—Regularity—Specifying Measures on the Line—Specifying Measures in R^k—Strange Euclidean Sets**

13. Measurable Functions and Mappings, 182

Measurable Mappings—Mappings into R^k—Limits and Measureability—Transformations of Measures

14. Distribution Functions, 187

Distribution Functions—Exponential Distributions—Weak Convergence—Convergence of Types—Extremal Distributions**

CHAPTER 3. INTEGRATION 199

15. The Integral, 199

Definition—Nonnegative Functions—Uniqueness

16. Properties of the Integral, 206

Equalities and Inequalities—Integration to the Limit—Integration over Sets—Densities—Change of Variable—Uniform Integrability—Complex Functions

17. The Integral with Respect to Lebesgue Measure, 221

The Lebesgue Integral on the Line—The Riemann Integral—The Fundamental Theorem of Calculus—Change of Variable—The Lebesgue Integral in R^k—Stieltjes Integrals

18. Product Measure and Fubini's Theorem, 231

Product Spaces—Product Measure—Fubini's Theorem—Integration by Parts—Products of Higher Order

19. The L^p Spaces,* 241

Definitions—Completeness and Separability—Conjugate Spaces—Weak Compactness—Some Decision Theory—The Space L^2—An Estimation Problem

CHAPTER 4. RANDOM VARIABLES AND EXPECTED VALUES 254

20. Random Variables and Distributions, 254

*Random Variables and Vectors—Subfields—Distributions—Multidimensional Distributions—Independence—Sequences of Random Variables—Convolution—Convergence in Probability—The Glivenko–Cantelli Theorem**

21. Expected Values, 273

Expected Value as Integral—Expected Values and Limits—Expected Values and Distributions—Moments—Inequalities—Joint Integrals—Independence and Expected Value—Moment Generating Functions

22. Sums of Independent Random Variables, 282

*The Strong Law of Large Numbers—The Weak Law and Moment Generating Functions—Kolmogorov's Zero–One Law—Maximal Inequalities—Convergence of Random Series—Random Taylor Series**

23. The Poisson Process, 297

Characterization of the Exponential Distribution—The Poisson Process—The Poisson Approximation—Other Characterizations of the Poisson Process—Stochastic Processes

24. The Ergodic Theorem,* 310

Measure-Preserving Transformations—Ergodicity—Ergodicity of Rotations—Proof of the Ergodic Theorem—The Continued-Fraction Transformation—Diophantine Approximation

CHAPTER 5. CONVERGENCE OF DISTRIBUTIONS 327

25. Weak Convergence, 327

Definitions—Uniform Distribution Modulo 1—Convergence in Distribution—Convergence in Probability—Fundamental Theorems—Helly's Theorem—Integration to the Limit*

26. Characteristic Functions, 342

*Definition—Moments and Derivatives—Independence—Inversion and the Uniqueness Theorem—The Continuity Theorem—Fourier Series**

27. The Central Limit Theorem, 357

*Identically Distributed Summands—The Lindeberg and Lyapounov Theorems—Dependent Variables**

28. Infinitely Divisible Distributions,* 371

Vague Convergence—The Possible Limits—Characterizing the Limit

29. Limit Theorems in R^k, 378

The Basic Theorems—Characteristic Functions—Normal Distributions in R^k—The Central Limit Theorem

30. The Method of Moments,* 388

The Moment Problem—Moment Generating Functions—Central Limit Theorem by Moments—Application to Sampling Theory—Application to Number Theory

CHAPTER 6. DERIVATIVES AND CONDITIONAL PROBABILITY 400

31. Derivatives on the Line,* 400

The Fundamental Theorem of Calculus—Derivatives of Integrals —Singular Functions—Integrals of Derivatives—Functions of Bounded Variation

32. The Radon–Nikodym Theorem, 419

Additive Set Functions—The Hahn Decomposition—Absolute Continuity and Singularity—The Main Theorem

33. Conditional Probability, 427

The Discrete Case—The General Case—Properties of Conditional Probability—Difficulties and Curiosities—Conditional Probability Distributions

34. Conditional Expectation, 445

Definition—Properties of Conditional Expectation—Conditional Distributions and Expectations—Sufficient Subfields—Minimum-Variance Estimation**

35. Martingales, 458

*Definition–Submartingdales—Gambling—Functions of Martingales—Stopping Times—Inequalities—Convergence Theorems—Applications: Derivatives—Likelihood Ratios— Reversed Martingales—Applications: de Finetti's Theorem— Bayes Estimation—A Central Limit Theorem**

CHAPTER 7. STOCHASTIC PROCESSES 482

36. Kolmogorov's Existence Theorem, 482

Stochastic Processes—Finite-Dimensional Distributions—Product Spaces—Kolmogorov's Existence Theorem—The Inadequacy of $\mathscr{R}^T$—A Return to Ergodic Theory—The Hewitt–Savage Theorem**

37. Brownian Motion, 498

Definition—Continuity of Paths—Measurable Processes— Irregularity of Brownian Motion Paths—The Strong Markov Property—The Reflection Principle—Skorohod Embedding— Invariance**

38. Nondenumerable Probabilities,* 526

Introduction—Definitions—Existence Theorems—Consequences of Separability

APPENDIX 536

NOTES ON THE PROBLEMS 552

BIBLIOGRAPHY 581

LIST OF SYMBOLS 585

INDEX 587