

Ordinary Differential Equations: Basics and Beyond

This book develops the theory of ordinary differential equations (ODEs), starting from an introductory level through to a graduate-level treatment of the qualitative theory, including bifurcation theory (but not chaos). While proofs are rigorous, the exposition is reader-friendly, aiming for the informality of face-to-face interactions.

A unique feature of this book is the integration of rigorous theory with numerous applications of scientific interest. Besides providing motivation, this synthesis clarifies the theory and enhances scientific literacy. Other features include: (i) a wealth of exercises at various levels, along with commentary that explains why they matter and (ii) a dedicated website with software templates, problem solutions, and other resources supporting the text and (iii) a large number and variety of illustrations. Given its many applications, the book is suitable for senior undergraduates and graduate students in mathematics and science and engineering courses. The thoughtful presentation, which anticipates many confusions of beginning students, makes the book suitable for a teaching environment that emphasizes self-directed, active learning (including the so-called inverted classroom).

Mathematics

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1	Introduction	1
1.1	Some Simple ODEs	1
1.1.1	Examples	1
1.1.2	Descriptive Concepts	2
1.2	Solutions of ODEs	4
1.2.1	Examples and Discussion	4
1.2.2	Geometric Interpretation of Solutions	5
1.3	Snippets of Solution Techniques	6
1.3.1	Computer Solutions	6
1.3.2	Linear Equations with Constant Coefficients	7
1.3.3	First-Order Linear Equations	8
1.3.4	Separable Equations	8
1.4	Physically Based Second-Order ODEs	10
1.4.1	Linear Spring–Mass Systems	10
1.4.2	Nonlinearity, Part I: The Restoring Force	12
1.4.3	Energy	14
1.4.4	Nonlinearity, Part II: The Frictional Force	17

1.5	Systems of ODEs	17
1.6	Topics Covered in This Book	20
1.6.1	General Overview	20
1.6.2	A Case Study in the Qualitative Theory of ODEs	21
1.7	Exercises	24
1.7.1	Core Exercises	26
1.7.2	Computational Exercises	29
1.7.3	Anticipatory Exercises	31
1.7.4	PHD Exercises	33
1.8	Pearls of Wisdom	36
1.8.1	Miscellaneous	36
1.8.2	The Concept “Generic”	37
1.8.3	A Comparison Theorem	38
2	Linear Systems with Constant Coefficients	41
2.1	Preview	41
2.2	Definition and Properties of the Matrix Exponential	42
2.2.1	Preliminaries About Norms	42
2.2.2	Convergence	46
2.2.3	The Main Theorem and Its Proof	48
2.3	Calculation of the Matrix Exponential	50
2.3.1	The Role of Similarity	50
2.3.2	Two Problematic Cases	52
2.3.3	Use of the Jordan Form	55

2.4	Large-Time Behavior of Solutions of Linear Systems	57
2.4.1	The Main Results	57
2.4.2	Tests for Eigenvalues in the Left Half-Plane	59
2.5	Classification and Phase Portraits for 2×2 Systems	60
2.5.1	Saddles and Nodes	60
2.5.2	Foci and Centers	62
2.5.3	Additional Remarks	62
2.6	Solution of Inhomogeneous Problems	63
2.7	Exercises	64
2.7.1	Core Exercises	64
2.7.2	Practice with Linear Algebra	68
2.7.3	Practice Sketching Phase Portraits	70
2.7.4	PHD Exercises	72
2.8	Pearls of Wisdom	74
2.8.1	Alternative Norms	74
2.8.2	Nondifferentiable Limits and the Cantor Set	75
2.8.3	More on Generic Behavior	77
	Nonlinear Systems: Local Theory	79
3.1	Two Counterexamples	79
3.2	Local Existence Theory	81
3.2.1	Statement of the Existence Theorem	81
3.2.2	$\mathcal{C}^1$ Implies Lipschitz Continuity	82
3.2.3	Reformulation of the IVP as an Integral Equation	85

3.2.4	The Contraction-Mapping Principle	85
3.2.5	Proof of the Existence Theorem	87
3.2.6	An Illustrative Example and Picard Iteration	90
3.2.7	Concluding Remarks	91
3.3	Uniqueness Theory	91
3.3.1	Gronwall's Lemma	91
3.3.2	More on Lipschitz Functions	93
3.3.3	The Uniqueness Theorem	94
3.4	Generalization to Nonautonomous Systems	96
3.4.1	Nonlinear Systems	96
3.4.2	Linear Systems	96
3.5	Exercises	97
3.5.1	Core Exercises	97
3.5.2	Linear ODEs with Periodic Coefficients	102
3.5.3	PHD Exercises	104
3.6	Pearls of Wisdom	106
3.6.1	Miscellaneous	106
3.6.2	Resonance	108
4	Nonlinear Systems: Global Theory	111
4.1	The Maximal Interval of Existence	112
4.2	Two Sufficient Conditions for Global Existence	113
4.2.1	Linear Growth of the RHS	113
4.2.2	Trapping Regions	114

4.3	Level Sets and Trapping Regions	119
4.3.1	Introduction via Duffing's Equation	119
4.3.2	The Chemostat	119
4.3.3	The Torqued Pendulum and ODEs on Manifolds	120
4.4	Nullclines and Trapping Regions	123
4.4.1	Nullclines in the Chemostat	123
4.4.2	An Activator–Inhibitor System	124
4.4.3	Sel'kov's Model for Glycolysis	127
4.4.4	Van der Pol's Equation	128
4.4.5	Michaelis–Menten Kinetics	129
4.5	Continuity Properties of the Solution	131
4.5.1	The Main Issue: Continuous Dependence on Initial Conditions	131
4.5.2	Some Associated Formalism	133
4.5.3	Continuity with Respect to Parameters	134
4.6	Differentiability Properties of the Solution	134
4.6.1	Dependence on Initial Conditions	134
4.6.2	The Perspective of Differentiability	136
4.6.3	Examples	137
4.6.4	The Order Notation	138
4.6.5	Proof of Theorem 4.6.1	140
4.6.6	Tying Up Loose Ends	141
4.6.7	Generalizations	141

4.3	Level Sets and Trapping Regions	119
4.3.1	Introduction via Duffing's Equation	119
4.3.2	The Chemostat	119
4.3.3	The Torqued Pendulum and ODEs on Manifolds	120
4.4	Nullclines and Trapping Regions	123
4.4.1	Nullclines in the Chemostat	123
4.4.2	An Activator–Inhibitor System	124
4.4.3	Sel'kov's Model for Glycolysis	127
4.4.4	Van der Pol's Equation	128
4.4.5	Michaelis–Menten Kinetics	129
4.5	Continuity Properties of the Solution	131
4.5.1	The Main Issue: Continuous Dependence on Initial Conditions	131
4.5.2	Some Associated Formalism	133
4.5.3	Continuity with Respect to Parameters	134
4.6	Differentiability Properties of the Solution	134
4.6.1	Dependence on Initial Conditions	134
4.6.2	The Perspective of Differentiability	136
4.6.3	Examples	137
4.6.4	The Order Notation	138
4.6.5	Proof of Theorem 4.6.1	140
4.6.6	Tying Up Loose Ends	141
4.6.7	Generalizations	141

4.7	Exercises	142
4.7.1	Core Exercises	143
4.7.2	Applying the Differentiation Theorems	149
4.7.3	Some Mopping-Up Exercises	151
4.7.4	Computing Exercise	152
4.7.5	PHD Exercises	153
4.8	Pearls of Wisdom	155
4.9	Appendix: Euler's Method	156
4.9.1	Introduction	156
4.9.2	Theoretical Basis for the Approximation	157
4.9.3	Convergence of the Numerical Solution	158
5	Nondimensionalization and Scaling	161
5.1	Classes of ODEs in Applications	162
5.1.1	Mechanical Models	162
5.1.2	Electrical Models	163
5.1.3	"Bathtub" Models	163
5.2	Scaling Example 0: Two Models from Ecology	165
5.3	Scaling Example 1: A Nonlinear Oscillator	169
5.4	Scaling Example 2: Sel'kov's Model for Glycolysis	172
5.5	Scaling Example 3: The Chemostat	174
5.5.1	ODEs Modeling Flow Through a Reactor	174
5.5.2	The Chemostat	176
5.6	Scaling Example 4: An Activator-Inhibitor System	178

5.7	Scaling Example 5: Michaelis–Menten Kinetics	181
5.8	Exercises	184
5.8.1	Core Exercises	185
5.8.2	Anticipatory Exercises	187
5.8.3	PHD Exercises	189
5.9	Pearls of Wisdom	191
5.9.1	Making Scaling Work for You	191
5.9.2	A Nod to Scientific Literacy	194
6	Trajectories Near Equilibria	195
6.1	Stability of Equilibria	196
6.1.1	The Main Theorem	196
6.1.2	An Easy Application	198
6.2	Terminology to Classify Equilibria	199
6.2.1	Terms Related to Theorem 6.1.1	199
6.2.2	Other Terms Based on Eigenvalues	201
6.2.3	Section 1.6 Revisited, Part I	202
6.2.4	Two-Dimensional Equilibria and Slopes of Nullclines	205
6.3	Activator–Inhibitor Systems and the Turing Instability	206
6.3.1	Equilibria of the Activator–Inhibitor System	206
6.3.2	The Turing Instability: Destabilization by Diffusion	208
6.4	Feedback Stabilization of an Inverted Pendulum	210
6.5	Lyapunov Functions	215
6.5.1	The Main Results	215

- 6.5.2 Lasalle’s Invariance Principle 217
 - 6.5.3 Construction of Lyapunov Functions: An Example 218
- 6.6 Stable and Unstable Manifolds 220
 - 6.6.1 A Linear Example 220
 - 6.6.2 Statement of the Local Theorem 222
 - 6.6.3 A Nonlinear Example 223
 - 6.6.4 Global Behavior of Stable/Unstable Manifolds 226
- 6.7 Drawing Phase Portraits 228
 - 6.7.1 Example 1: The Chemostat 229
 - 6.7.2 Example 2: The Activator–Inhibitor 230
 - 6.7.3 Example 3: Section 1.6 Revisited, Part II 231
- 6.8 Exercises 233
 - 6.8.1 Core Exercises 233
 - 6.8.2 Uses of a Lyapunov Function 238
 - 6.8.3 Phase-Portrait Exercises 240
 - 6.8.4 PHD Exercises 241
- 6.9 Pearls of Wisdom 245
 - 6.9.1 Miscellaneous 245
 - 6.9.2 The Hartman–Grobman Theorem
and Topological Conjugacy 246
 - 6.9.3 Structural Stability 247
- 6.10 Appendix 1: Partial Proof of Theorem 6.6.1 248
 - 6.10.1 Reformulation of the IVP as an Integral Equation 248
 - 6.10.2 Fixed-Point Analysis 250

6.10.3	The Stable Manifold	252
6.10.4	Stable Manifolds at Nonhyperbolic Equilibria	253
6.11	Appendix 2: Center Manifolds and Nonhyperbolicity	254
6.11.1	First Example	255
6.11.2	Second Example	256
7	Oscillations in ODEs	259
7.1	Periodic Solutions	259
7.1.1	Basic Issues	259
7.1.2	Examples of Periodic Solutions	261
7.1.3	A Leisurely Overview of This Chapter	265
7.2	Special Behavior in Two Dimensions ... Mostly	267
7.2.1	The Poincaré–Bendixson Theorem: Minimal Version	267
7.2.2	Applications of the Theorem	268
7.2.3	Limit Sets (in Any Dimension)	270
7.2.4	The Poincaré–Bendixson Theorem: Strong Version	274
7.2.5	Nonexistence: Dulac’s Theorem	275
7.2.6	Section 1.6 Revisited, Part III	275
7.3	Stability of Periodic Orbits and the Poincaré Map	276
7.3.1	An Eigenvalue Test for Stability	276
7.3.2	Basics of the Poincaré Map	278
7.3.3	Discrete-Time Dynamics and the Proof of Theorem 7.3.2	281
7.4	Stability of the Limit Cycle in the Torqued Pendulum	283

7.5	Van der Pol with Small β : Weakly Nonlinear Analysis	285
7.5.1	Two Illustrative Examples of Perturbation Theory	286
7.5.2	Application to the van der Pol Equation	291
7.6	Van der Pol with Large β : Singular Perturbation Theory	293
7.6.1	Two Sources of Guidance	293
7.6.2	Approximation of the Initial Decay in (7.58)	295
7.6.3	Phase-Plane Analysis of a Related Equation	298
7.6.4	Concluding Remarks	301
7.7	Stability of the van der Pol Limit Cycles	301
7.7.1	Case 1: Small β	302
7.7.2	Case 2: Large β	303
7.8	Exercises	304
7.8.1	Core Exercises	304
7.8.2	The Poincaré–Lindstedt Method	311
7.8.3	Changes of Variables	313
7.8.4	PHD Exercises	316
7.9	Pearls of Wisdom	318
7.9.1	Area and Dulac’s Theorem	318
7.9.2	Poincaré-Like Maps in Constructing Periodic Solutions	319
7.9.3	Stable/Unstable Manifolds in Other Contexts	320
7.9.4	Miscellaneous	321
7.10	Appendix: Stabilizing an Inverted Pendulum	321
7.10.1	A Smidgen of Floquet Theory	321

7.10.2	Some Stable Solutions of Mathieu's Equation	323
7.10.3	Application to the Inverted Vibrated Pendulum	325

8 Bifurcation from Equilibria **327**

8.1	Examples of Pitchfork Bifurcation	328
8.1.1	Bead on a Rotating Hoop	328
8.1.2	The Lorenz Equations	330
8.1.3	A Laterally Supported Inverted Pendulum	332
8.2	Perspectives on This Chapter	334
8.2.1	An Outline of the Chapter	334
8.2.2	A Bifurcation Theorem	334
8.3	Examples of Transcritical Bifurcation	336
8.3.1	Section 1.6 Revisited: Part IV	336
8.3.2	The Chemostat	337
8.4	Examples of Saddle-Node Bifurcation	339
8.4.1	The Torqued Pendulum	339
8.4.2	Activator-Inhibitor Systems	340
8.5	The Lyapunov-Schmidt Reduction	341
8.5.1	Bare Bones of the Reduction	341
8.5.2	Proof of Theorem 8.2.2	344
8.5.3	One-Dimensional Bifurcation Problems	348
8.5.4	Exchange of Stability	350
8.5.5	Symmetry and the Pitchfork Bifurcation	351
8.5.6	Additional Parameters in Bifurcation Problems	352

8.6	Steady-State Bifurcation in Two Applications	355
8.6.1	The Two-Cell Turing Instability	355
8.6.2	The CSTR	360
8.7	Examples of Hopf Bifurcation	364
8.7.1	An Academic Example	364
8.7.2	The “Repressilator”	366
8.7.3	Section 1.6 Revisited: Part V	368
8.7.4	The “Denatured” Morris–Lecar System	369
8.8	Theoretical Description of Hopf Bifurcation	372
8.8.1	A Bifurcation Theorem	372
8.8.2	The Activator–Inhibitor: Extreme Nongeneric Behavior	374
8.8.3	Sub/Supercriticality in Two Dimensions	375
8.9	Exercises	376
8.9.1	Core Exercises	376
8.9.2	Applications of Bifurcation Theory	380
8.9.3	PHD Exercises	385
8.10	Pearls of Wisdom	395
8.10.1	Comments on Proving the Hopf Bifurcation Theorem	395
8.10.2	High-Dimensional Bifurcation: Symmetry and Mode Competition	397
8.10.3	Homeostasis, or “Antibifurcation”	399

9 Examples of Global Bifurcation 403

9.1	Homoclinic Bifurcation	403
9.1.1	An Academic Example	403

9.1.2	The van der Pol Equation with a Nonlinear Restoring Force	405
9.1.3	Section 1.6 Revisited: Part VI	405
9.1.4	Other Examples of Homoclinic and Heteroclinic Bifurcations .	408
9.2	Saddle-Node Bifurcation of Limit Cycles	409
9.2.1	An Academic Example	409
9.2.2	The Denatured Morris–Lecar Equation	409
9.2.3	The Overdamped Torqued Pendulum	411
9.3	Poincaré Maps and Stability Loss of Limit Cycles	414
9.4	Mutual Annihilation of Two Limit Cycles	415
9.4.1	An Academic Example	415
9.4.2	The Denatured Morris–Lecar Equation	415
9.4.3	Phase-Locking in Coupled Oscillators	417
9.5	Hopf-Like Bifurcation to an Invariant Torus	417
9.5.1	An Academic Example	417
9.5.2	The Periodically Forced van der Pol Equation	419
9.5.3	Other Examples of Bifurcation to an Invariant Torus	420
9.6	Period-Doubling	420
9.6.1	Academic Example 1: Mappings	422
9.6.2	Cardiac Alternans	423
9.6.3	Academic Example 2: ODEs	426
9.6.4	A Periodically Forced Pendulum	428
9.6.5	Rössler’s Equation	430
9.6.6	Other Examples	432

9.7	The Onset of Chaos in the Lorenz Equations	433
9.8	Bursting in the Denatured Morris–Lecar Equations	438
9.9	Exercises	440
9.9.1	Core Exercises	440
9.9.2	Computations to Support Claims in the Text	444
9.9.3	Bifurcation in a Quadratic Map	446
9.9.4	PHD Exercises	447
9.10	Pearls of Wisdom	448
9.10.1	Remarks on Heteroclinic Orbits	448
9.10.2	Bifurcation in Fluid-Mechanics Problems	449
9.10.3	Routes to Chaos	449
10	Epilogue	451
10.1	Boundary Value Problems	451
10.1.1	An Overview Through Examples	451
10.1.2	Eigenvalue Problems	455
10.2	Stochastic Population Models	457
10.3	Numerical Methods: Two Sobering Examples	460
10.3.1	Stiff ODEs	460
10.3.2	Unreasonable Behavior of Reasonable Methods	461
10.4	ODEs on a Torus: Entrainment	463
10.5	Delay Differential Equations	467

10.6	A Peek at Chaos	470
10.6.1	A One-Dimensional Mapping Model	470
10.6.2	The Lorenz Equations	473
10.7	Exercises	479
10.7.1	Core Exercises	479
10.7.2	PHD Exercises	482
10.8	Pearls of Wisdom	484
10.8.1	The Elastica	484
10.8.2	A Bit More on Chaos	486
A Guide to Commonly Used Notation		487
A.1	Letter Choices	487
A.2	Other Notations	490
A.3	Other Conventions	491
B Notions from Advanced Calculus		493
B.1	Basic Issues	493
B.2	Pointwise and Uniform Convergence	495
B.2.1	Sequences	495
B.2.2	Series	497
B.2.3	Convergence of Integrals	499
B.3	Selected Issues in Vector Calculus	499
B.3.1	Differentiability	500
B.3.2	The Implicit Function Theorem	501
B.3.3	Surfaces and Manifolds	503

B.4	Exercises	506
B.4.1	Core Exercises	506
B.4.2	PHD Exercises	510
B.5	Pearls of Wisdom	512
C	Notions from Linear Algebra	515
C.1	How to Work with Jordan Normal Forms	515
C.2	The Real Canonical Form of a Matrix	519
C.3	Eigenvalues as Continuous Functions of Matrix Entries	520
C.4	The Routh–Hurwitz Criterion	522
C.5	Exercises	524
C.5.1	Core Exercises	524
C.5.2	PHD Exercises	526
	Bibliography	529
	Index	537