

Preface	xi
1 Background and Preview	1
1.1 The Real Numbers	2
1.2 Compact Sets of Real Numbers	8
1.3 Countable Sets	11
1.4 Uncountable Cardinals	15
1.5 Transfinite Ordinals	17
1.6 Category	21
1.7 Outer Measure and Outer Content	24
1.8 Small Sets	26
1.9 Measurable Sets of Real Numbers	30
1.10 Nonmeasurable Sets	35
1.11 Zorn's Lemma	38
1.12 Borel Sets of Real Numbers	41
1.13 Analytic Sets of Real Numbers	42
1.14 Bounded Variation	44
1.15 Newton's Integral	48
1.16 Cauchy's Integral	49
1.17 Riemann's Integral	51
1.18 Volterra's Example	54
1.19 Riemann–Stieltjes Integral	56
1.20 Lebesgue's Integral	60
1.21 The Generalized Riemann Integral	63
1.22 Additional Problems for Chapter 1	65
2 Measure Spaces	69
2.1 One-Dimensional Lebesgue Measure	70
2.2 Additive Set Functions	76
2.3 Measures and Signed Measures	82

2.4	Limit Theorems	86
2.5	Jordan and Hahn Decomposition	89
2.6	Complete Measures	93
2.7	Outer Measures	95
2.8	Method I	99
2.9	Regular Outer Measures	102
2.10	Nonmeasurable Sets	107
2.11	More About Method I	110
2.12	Completions	114
2.13	Additional Problems for Chapter 2	117
3	Metric Outer Measures	120
3.1	Metric Space	121
3.2	Metric Outer Measures	125
3.3	Method II	129
3.4	Approximations	133
3.5	Construction of Lebesgue–Stieltjes Measures	136
3.6	Properties of Lebesgue–Stieltjes Measures	142
3.7	Lebesgue–Stieltjes Measures in \mathbb{R}^n	147
3.8	Hausdorff Measures and Hausdorff Dimension	149
3.9	Methods III and IV	158
3.10	Additional Remarks	163
3.11	Additional Problems for Chapter 3	167
4	Measurable Functions	172
4.1	Definitions and Basic Properties	173
4.2	Sequences of Measurable Functions	179
4.3	Egoroff's Theorem	184
4.4	Approximations by Simple Functions	187
4.5	Approximation by Continuous Functions	192
4.6	Additional Problems for Chapter 4	197
5	Integration	201
5.1	Introduction	202
5.2	Integrals of Nonnegative Functions	206
5.3	Fatou's Lemma	211
5.4	Integrable Functions	215
5.5	Riemann and Lebesgue	219
5.6	Countable Additivity of the Integral	228
5.7	Absolute Continuity	231
5.8	Radon–Nikodym Theorem	237
5.9	Convergence Theorems	244

5.10	Relations to Other Integrals	251
5.11	Integration of Complex Functions	256
5.12	Additional Problems for Chapter 5	260
6	Fubini's Theorem	265
6.1	Product Measures	267
6.2	Fubini's Theorem	274
6.3	Tonelli's Theorem	277
6.4	Additional Problems for Chapter 6	278
7	Differentiation	281
7.1	The Vitali Covering Theorem	281
7.2	Functions of Bounded Variation	287
7.3	The Banach–Zarecki Theorem	291
7.4	Determining a Function by Its Derivative	295
7.5	Calculating a Function from Its Derivative	297
7.6	Total Variation of a Continuous Function	305
7.7	VBG _* Functions	310
7.8	Approximate Continuity, Lebesgue Points	315
7.9	Additional Problems for Chapter 7	321
8	Differentiation of Measures	328
8.1	Differentiation of Lebesgue–Stieltjes Measures	329
8.2	The Cube Basis; Ordinary Differentiation	333
8.3	The Lebesgue Decomposition Theorem	339
8.4	The Interval Basis; Strong Differentiation	342
8.5	Net Structures	349
8.6	Radon–Nikodym Derivative in a Measure Space	355
8.7	Summary, Comments, and References	364
8.8	Additional Problems for Chapter 8	368
9	Metric Spaces	370
9.1	Definitions and Examples	370
9.2	Convergence and Related Notions	379
9.3	Continuity	383
9.4	Homeomorphisms and Isometries	387
9.5	Separable Spaces	391
9.6	Complete Spaces	393
9.7	Contraction Maps	399
9.8	Applications of Contraction Mappings	401
9.9	Compactness	408
9.10	Totally Bounded Spaces	411

9.11	Compact Sets in $\mathcal{C}(X)$	413
9.12	Application of the Arzelà–Ascoli Theorem	417
9.13	The Stone–Weierstrass Theorem	419
9.14	The Isoperimetric Problem	423
9.15	More on Convergence	426
9.16	Additional Problems for Chapter 9	430
10	Baire Category	434
10.1	The Baire Category Theorem	434
10.2	The Banach–Mazur Game	441
10.3	The First Classes of Baire and Borel	446
10.4	Properties of Baire-1 Functions	452
10.5	Topologically Complete Spaces	456
10.6	Applications to Function Spaces	461
10.7	Additional Problems for Chapter 10	473
11	Analytic Sets	478
11.1	Products of Metric Spaces	479
11.2	Baire Space	481
11.3	Analytic Sets	484
11.4	Borel Sets	487
11.5	An Analytic Set That Is Not Borel	492
11.6	Measurability of Analytic Sets	494
11.7	The Suslin Operation	497
11.8	A Method to Show a Set Is Not Borel	498
11.9	Differentiable Functions	502
11.10	Additional Problems for Chapter 11	506
12	Banach Spaces	508
12.1	Normed Linear Spaces	509
12.2	Compactness	514
12.3	Linear Operators	518
12.4	Banach Algebras	523
12.5	The Hahn–Banach Theorem	526
12.6	Improving Lebesgue Measure	530
12.7	The Dual Space	537
12.8	The Riesz Representation Theorem	540
12.9	Separation of Convex Sets	546
12.10	An Embedding Theorem	550
12.11	The Uniform Boundedness Principle	553
12.12	An Application to Summability	556
12.13	The Open Mapping Theorem	561

12.14	The Closed Graph Theorem	565
12.15	Additional Problems for Chapter 12	567
13	The L_p Spaces	570
13.1	The Basic Inequalities	571
13.2	The ℓ_p and L_p Spaces ($1 \leq p < \infty$)	575
13.3	The Spaces ℓ_∞ and L_∞	577
13.4	Separability	580
13.5	The Spaces ℓ_2 and L_2	582
13.6	Continuous Linear Functionals	588
13.7	The L_p Spaces ($0 < p < 1$)	592
13.8	Relations	595
13.9	The Banach Algebra $L_1(\mathbb{R})$	598
13.10	Weak Sequential Convergence	604
13.11	Closed Subspaces of the L_p Spaces	606
13.12	Additional Problems for Chapter 13	610
14	Hilbert Spaces	612
14.1	Inner Products	613
14.2	Convex Sets	619
14.3	Continuous Linear Functionals	621
14.4	Orthogonal Series	624
14.5	Weak Sequential Convergence	630
14.6	Compact Operators	635
14.7	Projections	638
14.8	Eigenvectors and Eigenvalues	641
14.9	Spectral Decomposition	646
14.10	Additional Problems for Chapter 14	650
15	Fourier Series	653
15.1	Notation and Terminology	655
15.2	Dirichlet's Kernel	659
15.3	Fejér's Kernel	663
15.4	Convergence of the Cesàro Means	666
15.5	The Fourier Coefficients	670
15.6	Weierstrass Approximation Theorem	673
15.7	Pointwise Convergence: Jordan's Test	676
15.8	Pointwise Convergence: Dini's Test	681
15.9	Pointwise Divergence	683
15.10	Characterizations	686
15.11	Fourier Series in Hilbert Space	688
15.12	Riemann's Theorems	691

15.13 Cantor's Uniqueness Theorem	695
15.14 Additional Problems for Chapter 15	699
Index	701