

This first volume provides a modern introduction to Kählerian geometry and Hodge theory. It starts with basic material on complex variables, complex manifolds, holomorphic vector bundles, sheaves, and cohomology theory, the latter being treated in a more theoretical way than is usual in geometry, and culminates with the Hodge decomposition theorem. In between, the author proves the Kähler identities, which leads to the hard Lefschetz theorem and the Hodge index theorem.

The second part of the book investigates the meaning of these results in several directions. It introduces the notion of Hodge structure, the (logarithmic) de Rham complex, Frölicher spectral sequences, and mixed Hodge Structures. The book ends with a treatment of the deformations of the complex structure, Gauss–Manin connection, and variations of Hodge structure, on the one hand, and the study of algebraic cycles on the other. These topics will be further developed in the next volume.

The book is completely self-contained and can be used by students, either as a general introduction to complex algebraic geometry, or as a preparation for the second volume, which will present an up-to-date account of Hodge theory and algebraic cycles as developed by P. Griffiths and his school, by P. Deligne, and by S. Bloch. The text is complemented by exercises which provide further useful results in complex algebraic geometry.

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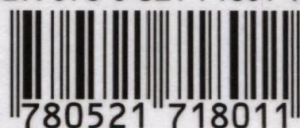
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