

PREFACE

xii

VOLUME ONE

1

1 PROPERTIES OF THE REAL NUMBERS

1

1.1	Introduction	1
1.2	The Real Number System	2
1.3	Algebraic Structure	4
1.4	Order Structure	7
1.5	Bounds	8
1.6	Sups and Infs	9
1.7	The Archimedean Property	12
1.8	Inductive Property of $\mathbb{N}$	13
1.9	The Rational Numbers Are Dense	14
1.10	The Metric Structure of $\mathbb{R}$	16
1.11	Challenging Problems for Chapter 1	18

2 SEQUENCES

21

2.1	Introduction	21
2.2	Sequences	22
2.2.1	Sequence Examples	23
2.3	Countable Sets	27
2.4	Convergence	29
2.5	Divergence	33
2.6	Boundedness Properties of Limits	35
2.7	Algebra of Limits	37
2.8	Order Properties of Limits	42
2.9	Monotone Convergence Criterion	47
2.10	Examples of Limits	50
2.11	Subsequences	54
2.12	Cauchy Convergence Criterion	58
2.13	Upper and Lower Limits	61
2.14	Challenging Problems for Chapter 2	66

iii

3	INFINITE SUMS	72
3.1	Introduction	72
3.2	Finite Sums	73
3.3	Infinite Unordered sums	78
3.3.1	Cauchy Criterion	79
3.4	Ordered Sums: Series	83
3.4.1	Properties	84
3.4.2	Special Series	85
3.5	Criteria for Convergence	90
3.5.1	Boundedness Criterion	91
3.5.2	Cauchy Criterion	91
3.5.3	Absolute Convergence	92
3.6	Tests for Convergence	95
3.6.1	Trivial Test	96
3.6.2	Direct Comparison Tests	96
3.6.3	Limit Comparison Tests	98
3.6.4	Ratio Comparison Test	99
3.6.5	d'Alembert's Ratio Test	100
3.6.6	Cauchy's Root Test	101
3.6.7	Cauchy's Condensation Test	103
3.6.8	Integral Test	104
3.6.9	Kummer's Tests	105
3.6.10	Raabe's Ratio Test	107
3.6.11	Gauss's Ratio Test	108
3.6.12	Alternating Series Test	110
3.6.13	Dirichlet's Test	111
3.6.14	Abel's Test	113
3.7	Rearrangements	117
3.7.1	Unconditional Convergence	118
3.7.2	Conditional Convergence	119
3.7.3	Comparison of $\sum_{i=1}^{\infty} a_i$ and $\sum_{i \in \mathbb{N}} a_i$	120
3.8	Products of Series	123
3.8.1	Products of Absolutely Convergent Series	125
3.8.2	Products of Nonabsolutely Convergent Series	126
3.9	Summability Methods	128
3.9.1	Cesàro's Method	128
3.9.2	Abel's Method	130
3.10	More on Infinite Sums	134
3.11	Infinite Products	135
3.12	Challenging Problems for Chapter 3	139
4	SETS OF REAL NUMBERS	146
4.1	Introduction	146

4.2	Points	147
4.2.1	Interior Points	147
4.2.2	Isolated Points	148
4.2.3	Points of Accumulation	149
4.2.4	Boundary Points	150
4.3	Sets	153
4.3.1	Closed Sets	153
4.3.2	Open Sets	154
4.4	Elementary Topology	159
4.5	Compactness Arguments	162
4.5.1	Bolzano-Weierstrass Property	163
4.5.2	Cantor's Intersection Property	164
4.5.3	Cousin's Property	166
4.5.4	Heine-Borel Property	168
4.5.5	Compact Sets	171
4.6	Countable Sets	173
4.7	Challenging Problems for Chapter 4	175
5	CONTINUOUS FUNCTIONS	179
5.1	Introduction to Limits	179
5.1.1	Limits (ϵ - δ Definition)	179
5.1.2	Limits (Sequential Definition)	183
5.1.3	Limits (Mapping Definition)	185
5.1.4	One-Sided Limits	186
5.1.5	Infinite Limits	188
5.2	Properties of Limits	190
5.2.1	Uniqueness of Limits	190
5.2.2	Boundedness of Limits	191
5.2.3	Algebra of Limits	192
5.2.4	Order Properties	195
5.2.5	Composition of Functions	198
5.2.6	Examples	200
5.3	Limits Superior and Inferior	205
5.4	Continuity	207
5.4.1	How to Define Continuity	207
5.4.2	Continuity at a Point	210
5.4.3	Continuity at an Arbitrary Point	213
5.4.4	Continuity on a Set	215
5.5	Properties of Continuous Functions	218
5.6	Uniform Continuity	219
5.7	Extremal Properties	222
5.8	Darboux Property	223
5.9	Points of Discontinuity	225

5.9.1	Types of Discontinuity	225
5.9.2	Monotonic Functions	227
5.9.3	How Many Points of Discontinuity?	230
5.10	Challenging Problems for Chapter 5	232
6	MORE ON CONTINUOUS FUNCTIONS AND SETS	239
6.1	Introduction	239
6.2	Dense Sets	239
6.3	Nowhere Dense Sets	241
6.4	The Baire Category Theorem	243
6.4.1	A Two-Player Game	243
6.4.2	The Baire Category Theorem	244
6.4.3	Uniform Boundedness	246
6.5	Cantor Sets	247
6.5.1	Construction of the Cantor Ternary Set	247
6.5.2	An Arithmetic Construction of K	250
6.5.3	The Cantor Function	252
6.6	Borel Sets	253
6.6.1	Sets of Type G_δ	254
6.6.2	Sets of Type F_σ	256
6.7	Oscillation and Continuity	257
6.7.1	Oscillation of a Function	258
6.7.2	The Set of Continuity Points	260
6.8	Sets of Measure Zero	263
6.9	Challenging Problems for Chapter 6	268
7	DIFFERENTIATION	271
7.1	Introduction	271
7.2	The Derivative	271
7.2.1	Definition of the Derivative	272
7.2.2	Differentiability and Continuity	276
7.2.3	The Derivative as a Magnification	277
7.3	Computations of Derivatives	278
7.3.1	Algebraic Rules	279
7.3.2	The Chain Rule	281
7.3.3	Inverse Functions	285
7.3.4	The Power Rule	286
7.4	Continuity of the Derivative?	288
7.5	Local Extrema	290
7.6	Mean Value Theorem	292
7.6.1	Rolle's Theorem	292
7.6.2	Mean Value Theorem	294
7.6.3	Cauchy's Mean Value Theorem	297

7.7	Monotonicity	298
7.8	Dini Derivates	300
7.9	The Darboux Property of the Derivative	304
7.10	Convexity	307
7.11	L'Hôpital's Rule	312
7.11.1	L'Hôpital's Rule: $\frac{0}{0}$ Form	313
7.11.2	L'Hôpital's Rule as $x \rightarrow \infty$	315
7.11.3	L'Hôpital's Rule: $\frac{\infty}{\infty}$ Form	316
7.12	Taylor Polynomials	319
7.13	Challenging Problems for Chapter 7	322
8	THE INTEGRAL	331
8.1	Introduction	331
8.2	Cauchy's First Method	333
8.2.1	Scope of Cauchy's First Method	335
8.3	Properties of the Integral	338
8.4	Cauchy's Second Method	343
8.5	Cauchy's Second Method (Continued)	345
8.6	The Riemann Integral	347
8.6.1	Some Examples	349
8.6.2	Riemann's Criteria	350
8.6.3	Lebesgue's Criterion	352
8.6.4	What Functions Are Riemann Integrable?	354
8.7	Properties of the Riemann Integral	356
8.8	The Improper Riemann Integral	359
8.9	More on the Fundamental Theorem of Calculus	361
8.10	Challenging Problems for Chapter 8	363
	VOLUME TWO	365
9	SEQUENCES AND SERIES OF FUNCTIONS	366
9.1	Introduction	366
9.2	Pointwise Limits	367
9.3	Uniform Limits	373
9.3.1	The Cauchy Criterion	375
9.3.2	Weierstrass M -Test	377
9.3.3	Abel's Test for Uniform Convergence	378
9.4	Uniform Convergence and Continuity	383
9.4.1	Dini's Theorem	385
9.5	Uniform Convergence and the Integral	387
9.5.1	Sequences of Continuous Functions	387
9.5.2	Sequences of Riemann Integrable Functions	389
9.5.3	Sequences of Improper Integrals	391

9.6	Uniform Convergence and Derivatives	393
9.6.1	Limits of Discontinuous Derivatives	394
9.7	Pompeiu's Function	397
9.8	Continuity and Pointwise Limits	399
9.9	Challenging Problems for Chapter 9	402
10	POWER SERIES	404
10.1	Introduction	404
10.2	Power Series: Convergence	404
10.3	Uniform Convergence	410
10.4	Functions Represented by Power Series	412
10.4.1	Continuity of Power Series	412
10.4.2	Integration of Power Series	413
10.4.3	Differentiation of Power Series	414
10.4.4	Power Series Representations	416
10.5	The Taylor Series	418
10.5.1	Representing a Function by a Taylor Series	420
10.5.2	Analytic Functions	422
10.6	Products of Power Series	424
10.6.1	Quotients of Power Series	425
10.7	Composition of Power Series	426
10.8	Trigonometric Series	428
10.8.1	Uniform Convergence of Trigonometric Series	428
10.8.2	Fourier Series	429
10.8.3	Convergence of Fourier Series	430
10.8.4	Weierstrass Approximation Theorem	433
11	THE EUCLIDEAN SPACES $\mathbb{R}^n$	437
11.1	The Algebraic Structure of $\mathbb{R}^n$	437
11.2	The Metric Structure of $\mathbb{R}^n$	439
11.3	Elementary Topology of $\mathbb{R}^n$	442
11.4	Sequences in $\mathbb{R}^n$	445
11.5	Functions and Mappings	448
11.5.1	Functions from $\mathbb{R}^n \rightarrow \mathbb{R}$	449
11.5.2	Functions from $\mathbb{R}^n \rightarrow \mathbb{R}^m$	451
11.6	Limits of Functions from $\mathbb{R}^n \rightarrow \mathbb{R}^m$	453
11.6.1	Definition	453
11.6.2	Coordinate-Wise Convergence	455
11.6.3	Algebraic Properties	458
11.7	Continuity of Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$	458
11.8	Compact Sets in $\mathbb{R}^n$	461
11.9	Continuous Functions on Compact Sets	462
11.10	Additional Remarks	463

12 DIFFERENTIATION ON $\mathbb{R}^n$	467
12.1 Introduction	467
12.2 Partial and Directional Derivatives	467
12.2.1 Partial Derivatives	469
12.2.2 Directional Derivatives	471
12.2.3 Cross Partial	473
12.3 Integrals Depending on a Parameter	476
12.4 Differentiable Functions	480
12.4.1 Approximation by Linear Functions	481
12.4.2 Definition of Differentiability	481
12.4.3 Differentiability and Continuity	485
12.4.4 Directional Derivatives	486
12.4.5 An Example	488
12.4.6 Sufficient Conditions for Differentiability	490
12.4.7 The Differential	492
12.5 Chain Rules	494
12.5.1 Preliminary Discussion	494
12.5.2 Informal Proof of a Chain Rule	497
12.5.3 Notation of Chain Rules	498
12.5.4 Proofs of Chain Rules (I)	500
12.5.5 Mean Value Theorem	502
12.5.6 Proofs of Chain Rules (II)	503
12.5.7 Higher Derivatives	505
12.6 Implicit Function Theorems	507
12.6.1 One-Variable Case	508
12.6.2 Several-Variable Case	511
12.6.3 Simultaneous Equations	514
12.6.4 Inverse Function Theorem	517
12.7 Functions From $\mathbb{R} \rightarrow \mathbb{R}^m$	520
12.8 Functions From $\mathbb{R}^n \rightarrow \mathbb{R}^m$	523
12.8.1 Review of Differentials and Derivatives	524
12.8.2 Definition of the Derivative	526
12.8.3 Jacobians	527
12.8.4 Chain Rules	530
12.8.5 Proof of Chain Rule	532
13 METRIC SPACES	537
13.1 Introduction	537
13.2 Metric Spaces—Specific Examples	539
13.3 Additional Examples	543
13.3.1 Sequence Spaces	543
13.3.2 Function Spaces	545
13.4 Convergence	548

13.5	Sets in a Metric Space	552
13.6	Functions	558
13.6.1	Continuity	560
13.6.2	Homeomorphisms	564
13.6.3	Isometries	570
13.7	Separable Spaces	573
13.8	Complete Spaces	575
13.8.1	Completeness Proofs	576
13.8.2	Subspaces of a Complete Space	578
13.8.3	Cantor Intersection Property	578
13.8.4	Completion of a Metric Space	579
13.9	Contraction Maps	581
13.10	Applications of Contraction Maps (I)	588
13.11	Applications of Contraction Maps (II)	591
13.11.1	Systems of Equations (Example 13.79 Revisited)	591
13.11.2	Infinite Systems (Example 13.80 revisited)	592
13.11.3	Integral Equations (Example 13.81 revisited)	594
13.11.4	Picard's Theorem (Example 13.82 revisited)	595
13.12	Compactness	597
13.12.1	The Bolzano-Weierstrass Property	597
13.12.2	Continuous Functions on Compact Sets	600
13.12.3	The Heine-Borel Property	602
13.12.4	Total Boundedness	604
13.12.5	Compact Sets in $C[a, b]$	607
13.12.6	Peano's Theorem	611
13.13	Baire Category Theorem	614
13.13.1	Nowhere Dense Sets	614
13.13.2	The Baire Category Theorem	617
13.14	Applications of the Baire Category Theorem	620
13.14.1	Functions Whose Graphs "Cross No Lines"	620
13.14.2	Nowhere Monotonic Functions	623
13.14.3	Continuous Nowhere Differentiable Functions	624
13.14.4	Cantor Sets	625
13.15	Challenging Problems for Chapter 13	627

VOLUME ONE **A-1**

A APPENDIX: BACKGROUND **A-1**

A.1	Should I Read This Chapter?	A-1
A.2	Notation	A-1
A.2.1	Set Notation	A-1
A.2.2	Function Notation	A-4
A.3	What Is Analysis?	A-9

A.4	Why Proofs?	A-10
A.5	Indirect Proof	A-11
A.6	Contraposition	A-12
A.7	Counterexamples	A-13
A.8	Induction	A-14
A.9	Quantifiers	A-17

SUBJECT INDEX

A-20