Contents

nage xiii Preface Ac

1

cjuc	-	pe	Noc Am
cknow	wledgem	nents	xvii
Ma	thematic	cal Preliminaries	1
1.1	Notatio	on and Preliminary Definitions	2
	1.1.1	Integers, Rationals, Reals, \mathbb{R}^n	2
	1.1.2	Inner Product, Norm, Metric	4
1.2	Sets ar	ad Sequences in \mathbb{R}^n	7
	1.2.1	Sequences and Limits	7
	1.2.2	Subsequences and Limit Points	10
	1.2.3	Cauchy Sequences and Completeness	11
	1.2.4	Suprema, Infima, Maxima, Minima	14
	1.2.5	Monotone Sequences in \mathbb{R}	17
	1.2.6	The Lim Sup and Lim Inf	18
	1.2.7	Open Balls, Open Sets, Closed Sets	22
	1.2.8	Bounded Sets and Compact Sets	23
	1.2.9	Convex Combinations and Convex Sets	23
	1.2.10	Unions, Intersections, and Other Binary Operations	24
1.3	Matric	es manoent aantaatow af T	30
	1.3.1	Sum, Product, Transpose	30
	1.3.2	Some Important Classes of Matrices	32
	1.3.3	Rank of a Matrix	33
	1.3.4	The Determinant	35
	1.3.5	The Inverse	38
	1.3.6	Calculating the Determinant	39
1.4	4 Functions		
	1.4.1	Continuous Functions	41
	1.4.2	Differentiable and Continuously Differentiable Functions	43

viii	Contents	
	1.4.3 Partial Derivatives and Differentiability	46
	1.4.4 Directional Derivatives and Differentiability	48
	1.4.5 Higher Order Derivatives	49
1 5	Quadratic Forms: Definite and Semidefinite Matrices	50
1	1.5.1 Quadratic Forms and Definiteness	50
	1.5.2 Identifying Definiteness and Semidefiniteness	53
1.6	Some Important Results	55
	1.6.1 Separation Theorems	56
	1.6.2 The Intermediate and Mean Value Theorems	60
	1.6.3 The Inverse and Implicit Function Theorems	65
1.7	7 Exercises	66
2 01	ptimization in \mathbb{R}^n	74
2.	Optimization Problems in \mathbb{R}^n	74
2.	2 Optimization Problems in Parametric Form	77
2.	3 Optimization Problems: Some Examples	78
	2.3.1 Utility Maximization	78
	2.3.2 Expenditure Minimization	79
	2.3.3 Profit Maximization	80
	2.3.4 Cost Minimization	80
	2.3.5 Consumption-Leisure Choice	81
	2.3.6 Portfolio Choice	81
	2.3.7 Identifying Pareto Optima	82
	2.3.8 Optimal Provision of Public Goods	83
	2.3.9 Optimal Commodity Taxation	84
2	4 Objectives of Optimization Theory	85
2	5 A Roadmap and 2 been Closed 2 man O allo B man O	86
2.	6 Exercises	88
3 E	xistence of Solutions: The Weierstrass Theorem	90
3.	1 The Weierstrass Theorem	90
3.	2 The Weierstrass Theorem in Applications	92
3.	3 A Proof of the Weierstrass Theorem	96
3.	4 Exercises citrate a to shall a	97
		100
4 U	Inconstrained Optima	100
4	1 "Unconstrained" Optima	100
4	2 First-Order Conditions	101
4	.3 Second-Order Conditions	103
4	4 Using the First- and Second-Order Conditions	104

		Contents	ix
	4.5	A Proof of the First-Order Conditions	106
	4.6	A Proof of the Second-Order Conditions	108
	4.7	Exercises	110
5	Equ	ality Constraints and the Theorem of Lagrange	112
	5.1	Constrained Optimization Problems	112
	5.2	Equality Constraints and the Theorem of Lagrange	113
		5.2.1 Statement of the Theorem	114
		5.2.2 The Constraint Qualification	115
		5.2.3 The Lagrangean Multipliers not served an and serve the transmission of transmission of the transmission of transmission of the transmission of t	116
	5.3	Second-Order Conditions	117
	5.4	Using the Theorem of Lagrange	121
		5.4.1 A "Cookbook" Procedure	121
		5.4.2 Why the Procedure Usually Works	122
		5.4.3 When It Could Fail	123
		5.4.4 A Numerical Example	127
	5.5	Two Examples from Economics	128
		5.5.1 An Illustration from Consumer Theory	128
		5.5.2 An Illustration from Producer Theory	130
		5.5.3 Remarks	132
	5.6	A Proof of the Theorem of Lagrange	135
	5.7	A Proof of the Second-Order Conditions	137
	5.8	Exercises and testimityO bas voicovarO-izerO 4.	142
6	Inec	uality Constraints and the Theorem of Kuhn and Tucker	145
	6.1	The Theorem of Kuhn and Tucker	145
		6.1.1 Statement of the Theorem	145
		6.1.2 The Constraint Qualification	147
		6.1.3 The Kuhn–Tucker Multipliers	148
	6.2	Using the Theorem of Kuhn and Tucker	150
		6.2.1 A "Cookbook" Procedure	150
		6.2.2 Why the Procedure Usually Works	151
		6.2.3 When It Could Fail	152
		6.2.4 A Numerical Example	155
	6.3	Illustrations from Economics	157
		6.3.1 An Illustration from Consumer Theory	158
		6.3.2 An Illustration from Producer Theory	161
	6.4	The General Case: Mixed Constraints	164
	6.5	A Proof of the Theorem of Kuhn and Tucker	165
	6.6	Exercises	168

7 Cor	vex Structures in Optimization Theory	172
7.1	Convexity Defined	173
	7.1.1 Concave and Convex Functions	174
	7.1.2 Strictly Concave and Strictly Convex Functions	176
7.2	Implications of Convexity	177
	7.2.1 Convexity and Continuity	177
	7.2.2 Convexity and Differentiability	179
	7.2.3 Convexity and the Properties of the Derivative	183
7.3	Convexity and Optimization	185
	7.3.1 Some General Observations	185
	7.3.2 Convexity and Unconstrained Optimization	187
	7.3.3 Convexity and the Theorem of Kuhn and Tucker	187
7.4	Using Convexity in Optimization	189
7.5	A Proof of the First-Derivative Characterization of Convexity	190
7.6	A Proof of the Second-Derivative Characterization of Convexity	191
7.7	A Proof of the Theorem of Kuhn and Tucker under Convexity	194
7.8	Exercises	198
8 Qua	si-Convexity and Optimization	203
8.1	Quasi-Concave and Quasi-Convex Functions	204
8.2	Quasi-Convexity as a Generalization of Convexity	205
8.3	Implications of Quasi-Convexity	209
8.4	Quasi-Convexity and Optimization	213
8.5	Using Quasi-Convexity in Optimization Problems	215
8.6	A Proof of the First-Derivative Characterization of Quasi-Convexity	216
8.7	A Proof of the Second-Derivative Characterization of	
	Quasi-Convexity	217
8.8	A Proof of the Theorem of Kuhn and Tucker under Ouasi-Convexity	220
8.9	Exercises enabled and and a second and a second and a second seco	221
9 Para	ametric Continuity: The Maximum Theorem	224
9.1	Correspondences	225
	9.1.1 Upper- and Lower-Semicontinuous Correspondences	225
	9.1.2 Additional Definitions	228
	9.1.3 A Characterization of Semicontinuous Correspondences	229
	9.1.4 Semicontinuous Functions and Semicontinuous	
	Correspondences	233
9.2	Parametric Continuity: The Maximum Theorem	235
	9.2.1 The Maximum Theorem	235
	9.2.2 The Maximum Theorem under Convexity	237

Contents

Х

			Contents		xi
	9.3	An App	lication to Consumer Theory		240
		9.3.1	Continuity of the Budget Correspondence		240
		9.3.2	The Indirect Utility Function and Demand		
			Correspondence		242
	9.4	An App	lication to Nash Equilibrium		243
		9.4.1	Normal-Form Games		243
		9.4.2	The Brouwer/Kakutani Fixed Point Theorem		244
		9.4.3	Existence of Nash Equilibrium		246
	9.5	Exercise	es		247
10	Supe	rmodula	rity and Parametric Monotonicity		253
	10.1	Lattice	s and Supermodularity		254
		10.1.1	Lattices		254
		10.1.2	Supermodularity and Increasing Differences		255
	10.2	Parame	etric Monotonicity		258
	10.3	An Ap	plication to Supermodular Games		262
		10.3.1	Supermodular Games		262
		10.3.2	The Tarski Fixed Point Theorem		263
		10.3.3	Existence of Nash Equilibrium		263
	10.4	A Proo	f of the Second-Derivative Characterization of		
		Supern	nodularity		264
	10.5	Exercis	ses		266
11	Finit	e-Horizo	on Dynamic Programming		268
	11.1	Dynam	nic Programming Problems		268
	11.2	Finite-	Horizon Dynamic Programming		268
	11.3	Histori	es, Strategies, and the Value Function		269
	11.4	Marko	vian Strategies		271
	11.5	Exister	nce of an Optimal Strategy		272
	11.6	An Ex	ample: The Consumption-Savings Problem		276
	11.7	Exerci	ses		278
12	Stat	ionary D	iscounted Dynamic Programming		281
	12.1	1 Description of the Framework			
	12.2	Histor	ies, Strategies, and the Value Function		282
	12.3	The B	ellman Equation		283
	12.4	A Tecl	hnical Digression		286
		12.4.1	Complete Metric Spaces and Cauchy Sequence	es	286
		12.4.2	Contraction Mappings		287
		12.4.3	Uniform Convergence		289

Contents

12.5	Existe	nce of an Optimal Strategy	29	91
	12.5.1	A Preliminary Result	29	92
	12.5.2	Stationary Strategies	29	94
	12.5.3	Existence of an Optimal Strategy	29	95
12.6	An Exa	ample: The Optimal Growth Model	29	98
	12.6.1	The Model	29	99
	12.6.2	Existence of Optimal Strategies	30	00
	12.6.3	Characterization of Optimal Strategies	30	01
12.7	Exercis	es Contraction assistants.	3(09
Appendi	x A Se	t Theory and Logic: An Introduction	3	15
A.1	Sets, U	nions, Intersections	3	15
A.2	Proposi	itions: Contrapositives and Converses	3	16
A.3	Quantif	fiers and Negation	3	18
A.4	Necess	ary and Sufficient Conditions	\$ 32	20
Appendi	x B Th	e Real Line	3:	23
B.1	Constru	ction of the Real Line	32	23
B.2	Propert	ies of the Real Line	32	26
Appendi	x C Str	ructures on Vector Spaces	3:	30
C.1	Vector	Spaces	33	30
C.2	Inner P	roduct Spaces	33	32
C.3	Norme	d Spaces	33	33
C.4	Metric	Spaces	33	36
	C.4.1	Definitions	33	36
	C.4.2	Sets and Sequences in Metric Spaces	33	37
	C.4.3	Continuous Functions on Metric Spaces	33	39
	C.4.4	Separable Metric Spaces	34	40
	C.4.5	Subspaces	34	41
C.5	Topolo	gical Spaces	34	42
	C.5.1	Definitions	34	42
	C.5.2	Sets and Sequences in Topological Spaces	34	43
	C.5.3	Continuous Functions on Topological Spaces	34	43
	C.5.4	Bases	34	43
C.6	Exercis	es	34	45
Bibliogra	aphy		34	49
Index	an bearing		35	51