

Semilinear Schrödinger Equations

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The nonlinear Schrödinger equation has received a great deal of attention from mathematicians, in particular because of its applications to nonlinear optics. It is also a good model dispersive equation, since it is often technically simpler than other dispersive equations, such as the wave or Korteweg–de Vries equation. Particularly useful tools in studying the nonlinear Schrödinger equation are energy and Strichartz's estimates.

This book presents various mathematical aspects of the nonlinear Schrödinger equation. It examines both problems of local nature (local existence of solutions, uniqueness, regularity, smoothing effects) and problems of global nature (finite-time blowup, global existence, asymptotic behavior of solutions). The methods presented apply in principle to a large class of dispersive semilinear equations. Basic notions of functional analysis (*Fourier transform, Sobolev spaces, etc.*) are recalled in the first chapter, but the book is otherwise mostly self-contained.



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