

Contents

I	THE PHYSICAL BACKGROUND	1
1.	Controlling the Quantum World	3
1.1	Quantum Optics	5
1.1.1	The Development of the Laser	5
1.1.2	The Development of Quantum Optics	6
1.1.3	The Elementary Objects of Ideal Quantum Optics	8
1.2	Quantum Information	9
1.2.1	The Einstein–Podolsky–Rosen Paradox	9
1.2.2	Schrödinger’s Cat and Entangled States	10
1.2.3	The Quantum Computer	10
1.2.4	Shor’s Algorithm and the Cirac–Zoller Ion Trap Quantum Computer	11
1.2.5	Quantum Computing with Trapped Ions	11
1.2.6	Engineered Systems for Quantum Optics	13
2.	Describing the Quantum World	14
2.1	Classical Stochastic Processes	15
2.1.1	Probabilities, Paths and Correlations	15
2.1.2	Markov Processes and the Conditional Probability	16
2.1.3	Measurement and What to Measure	16
2.1.4	Correlation Functions and Spectra	16
2.2	Theoretical Quantum Optics	17
2.2.1	Quantum Fields and “Stripped-Down” Quantum Electrodynamics	17
2.2.2	The Building Blocks of Quantum-Optical Systems	18
2.2.3	The Quantum-Mechanical Master Equation	18
2.2.4	The Input-Output Formalism	18
2.3	Quantum Stochastic Methods	19
2.3.1	Quantum Markov Description	19
2.4	Ultra-Cold Atoms	20
2.4.1	Field Operators and Hamiltonians	21

2.4.2 Methodologies for Ultra-Cold Atoms	22
--	----

II CLASSICAL STOCHASTIC METHODS

3. Physics in a Noisy World	27
3.1 Brownian Motion and the Thermal Origin of Noise	27
3.1.1 Equation of Motion for Brownian Motion	28
3.1.2 Markov Processes	28
3.1.3 Mathematical Description of a Markov Process	29
3.2 Brownian Motion, Friction, Noise and Temperature	29
3.2.1 The Langevin Equation	29
3.2.2 Solution of the Langevin Equation	30
3.2.3 Fluctuation-Dissipation Theorem	31
3.2.4 Diffusion as a Result of Brownian Motion	32
3.3 Measurement in a Fluctuating System	34
3.3.1 Quantities Most Commonly Considered	34
3.3.2 The Regression Theorem	35
3.3.3 Stationary Processes	36
3.3.4 Spectrum and Autocorrelation Function	36
3.3.5 Fourier Analysis of Fluctuating Functions	38
4. Stochastic Differential Equations	39
4.1 Ito Stochastic Differential Equation	39
4.1.1 Calculus of Stochastic Differential Equations	40
4.1.2 Change of Variables: Ito's Formula	41
4.2 The Fokker–Planck Equation	42
4.3 The Stratonovich Stochastic Differential Equation	42
4.3.1 Change of Variables for the Stratonovich Stochastic Differential Equation	43
4.3.2 Equivalent Stratonovich and Ito Stochastic Differential Equations	44
4.3.3 Comparison of Ito and Stratonovich Formalisms	44
4.4 Systems with Many Variables	45
4.4.1 Fokker–Planck Equations with Many Variables	45
4.4.2 Complex Variable Systems	45
4.5 Numerical Simulation of Stochastic Differential Equations	46
5. The Fokker–Planck Equation	47
5.1 Fokker–Planck Equation in One Dimension	47
5.1.1 Boundary Conditions	48
5.1.2 Deterministic Motion	50
5.1.3 The Wiener Process	51

5.2	Eigenfunctions of the Fokker–Planck Equation	51
5.2.1	Construction of Eigenfunctions	51
5.2.2	Expansion in Eigenfunctions	53
5.2.3	Eigenfunctions for the Ornstein–Uhlenbeck Process	53
5.3	Many-Variable Fokker–Planck Equations	54
5.3.1	Boundary Conditions	54
5.3.2	Stationary Solutions and Potential Conditions	55
6.	Master Equations and Jump Processes	57
6.1	The Master Equation	57
6.1.1	The Continuous Time Random Walk	59
6.1.2	The Poisson Process	61
6.1.3	The Random Telegraph Process	62
6.1.4	Example—Simulating Jumps in a Two-Level Atom	63
7.	Applications of Random Processes	66
7.1	The Ornstein–Uhlenbeck Process	66
7.1.1	Ornstein–Uhlenbeck Process in One Dimension	66
7.1.2	Many-Variable Ornstein–Uhlenbeck Process	68
7.1.3	Stationary Correlation Functions and Spectrum	69
7.2	Johnson Noise	70
7.3	Complex Variable Oscillator Processes	71
7.3.1	Line Broadening in a Random Frequency Oscillator	72
7.3.2	The Thermalized Oscillator	73
7.3.3	Equations for Phase and Amplitude	74
7.3.4	The van der Pol Laser Equation	75
8.	The Markov Limit	78
8.1	The White Noise Limit	78
8.1.1	Models for Non-White Noise	79
8.1.2	The Projector Formalism	79
8.1.3	The Limiting Fokker–Planck Equation	82
8.1.4	Evaluation of the Diffusion Coefficient in Terms of Correlation Functions	82
8.1.5	Summary	84
8.2	Interpretation and Generalizations of the White Noise Limit	84
8.2.1	Result for a General Stochastic Differential Equation	85
8.2.2	Result for General Non-Gaussian Noises	85
8.2.3	Time Dependence of Coefficients	85
8.2.4	Time Dependence of the Noise Equation	86
8.3	Linear Non-Markovian Stochastic Differential Equations	86
8.3.1	The Complex Oscillator with a Noisy Frequency	87
8.3.2	Approximation Methods for Multivariable Systems	89
8.3.3	Example—The Two-Dimensional Oscillator Driven by Random Telegraph Noise	91

8.3.4 Driving with Other Kinds of Noise	94
9. Adiabatic Elimination of Fast Variables	97
9.1 Slow and Fast Variables	97
9.1.1 A Simplified Laser Model	98
9.1.2 Stochastic Elimination of the Fast Variable	100
9.2 Other Applications of the Adiabatic Elimination Method	104
9.2.1 A Stochastic Model of Trapped Atoms	104
9.2.2 Motional Narrowing	106
III FIELDS, QUANTA AND ATOMS	
10. Ideal Bose and Fermi Systems	111
10.1 The Quantum Gas	111
10.1.1 Bosons	111
10.1.2 Fermions	112
10.1.3 The Hamiltonian and Total Number Operators	112
10.2 Thermal States	113
10.2.1 Canonical Ensemble	113
10.2.2 The Grand Canonical Ensemble	114
10.3 Fluctuations in the Ideal Bose Gas	115
10.3.1 Moments of the Number Operator	115
10.3.2 Many Modes	116
10.4 Bosonic Quantum Gaussian Systems	117
10.4.1 Hartree–Fock Factorization	117
10.4.2 Generalized Hartree–Fock Factorization for Quantum Gaussian Density Operators	118
10.5 Coherent States	119
10.5.1 Properties of the Coherent States	119
10.5.2 The Harmonic Oscillator	122
10.6 Fluctuations in Systems of Fermions	123
10.6.1 The Single-Mode System	123
10.6.2 Fermi–Gaussian Systems	123
10.6.3 Hartree–Fock Factorization in Systems of Fermions	124
10.7 Two-Level Systems and Pauli Matrices	124
10.7.1 Pauli Matrix Properties	125
11. Quantum Fields	127
11.1 Kinds of Quantum Field	127
11.1.1 Matter Wave Fields	127
11.1.2 Sound Waves	129
11.1.3 The Electromagnetic Field	130

11.1.4 Monochromatic Electromagnetic Waves	133
11.1.5 States of Quantized Fields	134
11.2 Coherence and Correlation Functions	135
11.2.1 Interference of Classical Waves	136
11.2.2 Quantum Interference	138
11.2.3 Summary—Phase and Interference	141
12. Atoms, Light and their Interaction	143
12.1 Interaction with the Quantized Radiation Field	144
12.1.1 Hamiltonian and Schrödinger Equation	144
12.1.2 The Two-Level Atom Approximation	147
12.1.3 The Rotating Wave Approximation	147
12.1.4 “Stripped-Down” Quantum Electrodynamics	148
12.2 Decay of an Excited Atom	148
12.2.1 Wavefunction and Initial Condition	148
12.2.2 Solutions for Atomic Decay and Radiated Field	152
12.3 The Two-Level Atom in a Strong Classical Driving Field	154
12.3.1 Interaction Hamiltonian	154
12.3.2 Solution of the Schrödinger Equation	155
12.3.3 Optical Pulses	156
12.3.4 Effective Potential on a Ground State Atom	158
12.4 Interaction of a Two-Level Atom with a Single Mode	159
12.4.1 Quantum Collapses and Revivals	160
IV QUANTUM MEASUREMENT THEORY	
IV QUANTUM STOCHASTIC PROCESSES	163
13. Quantum Markov Processes	163
13.1 Two-Level Atom in a Finite-Temperature Electromagnetic Field	163
13.1.1 The Quantum-Mechanical Master Equation	163
13.1.2 System and Heat Bath	164
13.1.3 The Master Equation	165
13.2 Derivation of the Master Equation	168
13.2.1 Description of Projection Method	168
13.2.2 Explicit Formulation as a Quantum Master Equation	172
13.2.3 Final Form of the Master Equation	174
13.3 More General Heat Baths	174
13.3.1 Generalized Bath Operators	175
13.4 Quantum Correlation Functions and Spectra	177
13.4.1 The Evolution Operator	177
13.4.2 Multitime Averages	177
13.4.3 Quantum Regression Theorem	178
13.4.4 Spectrum and Quantum Correlation Functions	179

14. Applications of the Master Equation	181
14.1 A Two-Level Atom Interacting with a Thermal Heat Bath	181
14.1.1 Frequency Shifts	182
14.1.2 Equations of Motion	183
14.1.3 Master Equation for the Occupation Numbers	184
14.1.4 Comparison with Classical Damping	185
14.2 The Two-Level Atom Driven by a Coherent Light Field	186
14.2.1 The Resonant Optical Bloch Equations	187
14.2.2 Correlation Functions and Spectrum	190
14.3 Master Equations for Harmonic Oscillator Systems	191
14.3.1 Damping and Gain with a Harmonic Oscillator	191
14.3.2 Formulation in Terms of Density Operator Matrix Elements .	194
14.3.3 The Phase-Damped Oscillator	195
14.3.4 Decoherence	197
14.4 A Simple Model of Laser Cooling	197
14.4.1 Formulation of the Model	198
14.4.2 Doppler Cooling	200
14.4.3 Radiation Pressure Force Exerted by a Weak Standing Wave .	201
14.4.4 Fluctuations	201
V PHASE SPACE METHODS	
15. Phase Space Representations for Bosons	207
15.1 The Quantum Characteristic Function	208
15.1.1 Normally Ordered Quantum Characteristic Function	208
15.1.2 Antinormally Ordered Quantum Characteristic Function .	208
15.1.3 Symmetrically Ordered Quantum Characteristic Function .	208
15.1.4 Relationship between the Different Quantum Characteristic Functions	209
15.1.5 Properties of the Quantum Characteristic Functions	209
15.2 Phase Space Representations of the Density Operator	210
15.2.1 Wigner Function	210
15.2.2 The P-Function	212
15.2.3 The Q-Function	213
15.2.4 Multitime Correlation Functions	213
16. Wigner Function Methods	215
16.1 Operator Correspondences and Equations of Motion	215
16.1.1 The Driven Harmonic Oscillator	216
16.1.2 The Anharmonic Oscillator	218
16.1.3 The Bogoliubov Hamiltonian	219
16.2 Damped and Driven Systems	220

16.2.1	The Harmonic Oscillator Including Loss or Gain	220
16.2.2	The Phase-Damped Harmonic Oscillator	224
16.3	The Wigner Distribution Function $f(x, p)$	225
16.3.1	The Wigner Distribution Function as a Quasiprobability	226
16.3.2	Operator Mappings for the Wigner Distribution Function	227
16.3.3	Motion in a Potential	229
16.3.4	A Free Particle in One Dimension	230
16.3.5	Quantum Brownian Motion	231
16.4	Quantum Fluctuations in Equations of Motion	232
17. P-Function Methods		234
17.1	Operator Correspondences and Equations of Motion	234
17.1.1	The Driven Harmonic Oscillator	235
17.1.2	The Anharmonic Oscillator	236
17.2	Damped and Driven Systems	236
17.2.1	The Harmonic Oscillator Including Loss or Gain	236
17.2.2	The Phase-Damped Harmonic Oscillator	238
17.3	The Laser	238
17.3.1	A Simple Laser Model	238
17.3.2	Implementation of the Laser Equation	241
17.3.3	Solutions of the Laser Equations	242
VI QUANTUM MEASUREMENT THEORY		
18. Foundations and Formalism of Quantum Measurement		249
18.1	Formulations of Quantum Mechanics	249
18.1.1	Fundamental Postulates	250
18.1.2	Status of the Postulates	250
18.2	Modelling a Measurement—Tracks in a Cloud Chamber	251
18.2.1	Excitation of an Atom	251
18.2.2	Interpretation	254
18.2.3	Entanglement	254
18.2.4	Interpretation as Collapse of the Wavefunction	255
18.2.5	The Essential Features	257
18.3	Formal Quantum Measurement Theory	257
18.3.1	Measurement Operators	258
18.3.2	Density Operator Formulation of Measurement Theory	259
18.3.3	Example—The Two-Level Atom	259
18.4	Multitime Measurements	260
18.4.1	Sequences of Measurements	260
18.4.2	Expression as a Correlation Function	262
18.4.3	General Correlation Functions	262

19. Continuous Measurements	263
19.1 Photon Counting	263
19.1.1 Master Equation for Continuous Measurement	264
19.1.2 Measurement as a Physical Process	264
19.2 Wavefunction Interpretation of Continuous Measurement	265
19.2.1 Wavefunction Evolution	265
19.2.2 The Stochastic Schrödinger Equation	266
19.2.3 Multiple Detection	267
19.3 Application to Matter Wave Interference	269
19.3.1 Interference of Independent Bose–Einstein Condensates	269
19.4 Damping of Quantum Coherence	271
19.4.1 Stochastic Schrödinger Equation Treatment	272
19.4.2 Interference and the Quantum Characteristic Function	273
19.4.3 The Linear Loss-Gain Model	274
19.4.4 The Stability and Robustness of Coherent States	277
19.5 The Emergence of the Macroscopic World	278
19.5.1 Modelling Quantum Measurement	278
19.5.2 Zurek’s Formulation of Quantum Measurement Theory	280
19.5.3 Relationship to van Kampen’s Formulation of Measurement Theory	280
20. The Quantum Zeno Effect	281
20.1 Theoretical Basis for the Quantum Zeno Effect	281
20.1.1 Connection with Continuous Measurement Theory	282
20.2 A Quantum Model of Trapped Atoms	284
20.2.1 Stationary State and Projectors	285
20.2.2 The Master Equation in the Strong Dissipation Limit	286
20.3 Quantum Zeno Effect for a Bose–Einstein Condensate	288
20.3.1 P-Representation Solution	288
20.3.2 Implementation of Fast Loss Mechanism	290
References	293
Author Index	297
Subject Index	299