

This is an excellent textbook on analysis and it has several unique features: Proofs of heat kernel estimates, the Nash inequality and the logarithmic Sobolev inequality are topics that are seldom treated on the level of a textbook. Best constants in several inequalities, such as Young's inequality and the logarithmic Sobolev inequality, are also included. A thorough treatment of rearrangement inequalities and competing symmetries appears in book form for the first time. There is an extensive treatment of potential theory and its applications to quantum mechanics, which, again, is unique at this level. Uniform convexity of L^p space is treated very carefully. The presentation of this important subject is highly unusual for a textbook. All the proofs provide deep insights into the theorems. This book sets a new standard for a graduate textbook in analysis.

—Shing-Tung Yau, Harvard University

For some number of years, Rudin's "Real and Complex", and a few other analysis books, served as the canonical choice for the book to use, and to teach from, in a first year grad analysis course. Lieb-Loss offers a refreshing alternative: It begins with a down-to-earth intro to measure theory, L^p and all that ... It aims at a wide range of essential applications, such as the Fourier transform, and series, inequalities, distributions, and Sobolev spaces—PDE, potential theory, calculus of variations, and math physics (Schrödinger's equation, the hydrogen atom, Thomas-Fermi theory ... to mention a few). The book should work equally well in a one-, or in a two-semester course. The first half of the book covers the basics, and the rest will be great for students to have, regardless of whether or not it gets to be included in a course.

—Palle E. T. Jorgensen, University of Iowa

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